Using N-Ary Multi-Modal Logics in Argumentation Frameworks to Reason about Ethics

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Abstract. Autonomous behaviors may raise ethical issues that agents must consider in their reasoning. Some approaches use deontic logics, while others consider a value-based argumentation framework. However, no work combines both modal logic and argumentation to reason about ethics. Hence, we propose a new argumentation framework where arguments are built from a n-ary multi-modal logic. It allows to express different kinds of operators, e.g. nullary choice or moral worth operators, dyadic deontic operators or mental states. However the standard attacks based on logical contradictions are no longer sufficient to catch an intuitive meaning for attacks. Hence, we enrich standard attacks by characterizing how oppositions between modal operators arise. Furthermore we show the standard logic-based attacks have a quasi-symmetry property, i.e. when an argument attacks another, this argument is necessarily attacked by another one. Our modal attacks do not have this property, which is highly relevant to decide a dilemma.
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1. Introduction

Autonomous behaviors may raise ethical issues that autonomous artificial agents must represent and integrate in their reasoning. For instance, reasoning on codes of conduct may be important for medical autonomous agents in order to deal with medical secrecy or respect for dignity. Therefore, many frameworks have been developed in order to design autonomous agents embedded with explicit ethical concepts [1–13]. Those approaches model some specific aspects of ethical reasoning, but do not capture entirely how human beings make ethical reasoning. Indeed, options are evaluated by associating them to a good or bad value with respect to their compliance with mores, i.e. values and usages (customs) of a group or a single person [14–17]. In some situations two options are supported by different moral worth, each bringing some regret given that the execution of both is impossible: it is a *moral dilemma* [18]. To solve them, human beings engage in *ethical reasoning* to search for arguments that will support some values (i.e. moral worth) considered as important in the situation [19].

In this article, we aim to model such an ethical reasoning process by considering different moral worth and confronting them to each other through an argumentation framework (AF) as has been proposed in [13, 20, 21], more precisely a value-based argumentation framework (VAF) [22]. In VAFs, arguments are associated with a value that may correspond to a moral worth. For instance, let us consider an autonomous vehicle in an emergency situation which reasons with two arguments. A first one may say "it is forbidden to bypass the speed limit" and thus may promote a *law* value. A second one may say "a dying passenger must arrive at the hospital as quickly as possible" and may promote a *life* value.

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As associations between arguments and values are abstracted in VAFs, we propose to ground them on modal logics to describe the valid arguments and their associated value. Indeed, reasoning according to values can be done with a deontic logic reasoning on ideal worlds in the name of a given value. However, ethics is also related to contexts and must deal with exceptions (e.g. it is forbidden to lie except when one's life is in danger). While the standard deontic logic cannot deal with conditional (i.e. contextual) obligations, dyadic deontic operators can in a certain way. That is why it may be useful to consider high arity deontic operator where several parameters represent different contexts, and one parameter represents the deontic obligation. For instance, triadic deontic operators may represent deontic obligations for a given context, knowing some exceptions. Finally, modal logic is also adapted to express knowledge and beliefs. In the previous example, another argument may say "the passenger is not dying as she suffers a shoulder dislocation". Such an argument does not promote any moral worth but will clearly attack the previous.

Thus, the main contributions of this article are twofold:

(1) We provide a sound and complete axiomatic system to reason about n-ary multi-modal logics. Such a general modal logic is based on the work of Blackburn *et al.* [23] where n-ary modal operators are allowed thanks to similarity types. The latter notion represents all modal operators used in the logic and their associated arities. However, while Blackburn *et al.* proposed a semantics for n-ary normal modal logics, they did not propose any general axiomatic system. That is why we present in this article such an axiomatic system, and we show that this logical system is strongly sound and strongly complete.

(2) We propose an argumentation framework where arguments and attack relations are semantically defined with this n-ary multi-modal logic. Since modal operators can qualify opposite statements while not being logically inconsistent, the standard attacks (e.g. rebuttal, undercut, defeater) based on logical contradictions are no longer sufficient to catch an intuitive meaning for attacks. That is why we enrich standard attacks with *modal-mappings*, which characterize how oppositions between modal operators arise. We show that standard logic-based attacks have a quasi-symmetry property, i.e. when an argument attacks another, this argument is necessarily attacked in return, while our modal attacks do not have this property, which is highly relevant to decide a dilemma.

This article is organized as follows. Section 2 presents a running example that we use throughout the article to illustrate our framework. Section 3 recalls preliminary notions on value-based argumentation, and on how to combine modal logic with argumentation. We then define the logical framework while instantiating it on the running example in Section 4. Section 5 provides the axiomatic system of our framework and its main properties. To facilitate reading, all proofs are given in Appendix A.2 to A.7. Finally, we define a value-based argumentation framework structured by our logical system in Section 6.

2. The running example

This example aims to model the ethical reasoning of an agent deciding which transportation mode she should take. We assume the values of the agent, her preferences, her different possible plans are known. We also assume that the agent only takes rational decisions based on her preferences. Example 1 introduces some formal elements for this running application that will be used in the sequel.

Example 1. *We consider a set of values, denoted $\mathcal{V}al$, such as an environmental value (v_{env}), an economical value (v_{eco}), a value to respect authorities (v_{aut}) and a value associated to health (v_{health}). We consider the agent has to decide between four choices represented by a set of symbols:*

- $\triangle_{car}$ *:= "go by car"*
- $\triangle_{p.t.}$ *:= "go by public transport"*
- $\triangle_{bike}$ *:= "go by bike"*
- $\triangle_{walk}$ *:= "go by foot"*

We denote by $\mathcal{P}$ a set of propositional atoms on information about the agent and the situation. $\mathcal{P}$ contains the atoms:

- has_{bike} := *"the user has a bike"*
- has_{car} := *"the user has a car"*
- $crisis_{health}$:= *"there is a health crisis"*
- *congestion := "there is a congestion"*
- *emergency := "there is an emergency"*

To model ethical judgement with a VAF, modal operators need to be used both to link deductive arguments with values, and to define a non symmetrical attack. For instance, if one argument concludes that the agent knows that there is a congestion, all arguments that would conclude that it saves time to go by car would be attacked by the former one. However, the converse should not be true since the context when it saves time to go by car is not true due to our knowledge of the congestion. Furthermore, VAF could be useful when two arguments attack each other with distinct supported values. For instance, let us suppose two arguments, i.e "the authorities forbid to take public transport when there is a health crisis" which is supported by the health value, and "the environmental value prescribes to take public transport when you do not have a bike" which is supported by the environmental value. Both arguments are in logical contradiction since they describe opposite ideals in different contexts. Both should attack each other and VAF can be helpful if we consider that the health sanitary value is more important than environmental sustainability.

3. Preliminary notions

We present here how logic can be used to model argumentation, and basic notions on value-based argumentation, and on how to combine modal logic with argumentation.

3.1. Argumentation as logic

Caminada and Gabbay [24] identify three ways to use modal logic to model argumentation: (1) a meta-level approach where possible worlds are arguments and atoms identify if an argument (a world) is accepted, (2) a object-level approach where arguments are the atoms of a logic, and (3) a mixed approach that uses one of the previous approaches to characterize extension semantics. For instance, [25] considers the set of worlds as the possible arguments and the accessibility relation between worlds as the attacks between arguments. [26] consider sets of arguments as sets of atoms, and attack relations as formulas. There are other approaches – such as in [27] – that use *propositional dynamic logic* to model bipolar argumentation frameworks.

However, acceptability semantics cannot be combined with usual indiscernability relations for representing mental states such as knowledge or beliefs since different elements of language – argumentation and logical formulas that talk about possibilities, beliefs or knowledge – are mixed. Indeed, those semantics are not be able to express arguments which are based on mental states of agents such as ("since agent i knows that ϕ i.e. $K_i\phi$ is true", "then ϕ is true"). The reason is that $K_i\phi$ is a formula whose truth depends on other accessible possible worlds and not on arguments. In this case we see that possible worlds cannot be considered as arguments.

We conclude that while these works use modal logic to reason about argumentation, they cannot express arguments based on mental states or normative modalities. Thus, we need to consider another approach that uses a logical mechanism to build an argumentation graph which can express mental states.

3.2. Value-based argumentation

Classically, a VAF extends the standard AF introduced by [28] with the concept of values, and preferences on values [22].

Definition 1 (VAF). *A* VAF *is a tuple* $\langle \mathcal{A}, \mathcal{R}, \mathcal{V}al, \mu, \succ \rangle$ *where:*

- $\mathcal{A}$ *is a nonempty set of arguments,*
- $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ *is a binary relation denoting direct attacks between arguments of* $\mathcal{A}$*,*

- $\mathcal{V}al$ *is a nonempty set of values,*
- $\mu : \mathcal{A} \rightarrow \mathcal{V}al$ *is a mapping that assigns each argument a value it promotes,*
- $\succ \subseteq \mathcal{V}al \times \mathcal{V}al$ *is a transitive relation denoting preferences*[1] *between values.*

Since direct attacks in a VAF do not use values or preferences, a standard notion of defeat relation characterizes a direct attack from argument A to B when B is not preferred to A with respect to their assigned values. To prevent from having any ambiguity with the defeat relation used in logic-based argumentation [29], we prefer to call this notion a *VAF-attack*. This relation is used to define *conflict-freeness* and *acceptability*.

Definition 2 (VAF-attack)**.** *Let* $\langle \mathcal{A}, \mathcal{R}, \mathcal{V}al, \mu, \succ \rangle$ *be a VAF. Argument* $A \in \mathcal{A}$ VAF-attacks *argument* $B \in \mathcal{A}$*, written* $A\ \mathcal{R}_{vaf}\ B$ *if, and only if,* $A\mathcal{R}B$ *i.e. A attacks B, and* $\neg(\mu(B) \succ \mu(A))$ *i.e. the value* $\mu(B)$ *is not preferred to the value* $\mu(A)$*.*

Definition 3 (Conflict-freeness and acceptability)**.** *Let* $\mathcal{R}_{vaf}$ *be a VAF attack on* $\langle \mathcal{A}, \mathcal{R}, \mathcal{V}al, \mu, \succ \rangle$*, and* $S \subseteq \mathcal{A}$*:*

- S *is a* conflict-free *set of arguments w.r.t.* $\mathcal{R}_{vaf}$ *if, and only if,* $\nexists A, B \in S$ *s.t.* $A\ \mathcal{R}_{vaf}\ B$*,*
- *For all* $A \in \mathcal{A}$*, A is acceptable w.r.t.* S *if, only if,* $\forall B \in \mathcal{A}$*,* $B\ \mathcal{R}_{vaf}\ A$*,* $\exists C \in S, C\ \mathcal{R}_{vaf}\ B$*.*

Characterizing the winning sets of arguments is done with extension semantics [28]. As there exists many different extensions, we only recall some of them.

Definition 4 (Extensions)**.** *Let* $S \subseteq \mathcal{A}$ *be a set of arguments.* S *is an* extension *which is:*

- admissible *if, and only if,* S *is conflict-free and all arguments* $A \in S$ *are acceptable with respect to* S*,*
- complete *if, and only if,* S *is admissible and contains all acceptable arguments with respect to* S*,*
- grounded *if, and only if,* S *is a minimal complete extension with respect to* $\subsetneq$*,*
- preferred *if, and only if,* S *is a maximal admissible extension with respect to* $\subsetneq$*,*
- stable *if, and only if,* S *is conflict-free, and* $\forall A \in \mathcal{A} \setminus S$*,* $\exists B \in S$ *s.t.* $B\ \mathcal{R}_{vaf}\ A$*.*

In the general case there is no consensus about which extension semantics to use. However as suggested in [30] some extensions can be considered as more preferable due to their uniqueness, e.g. the grounded or the ideal extension. For instance, [20] claim that the grounded semantics is the best suited one for two main reasons: (1) the uniqueness of the grounded semantics make it easy to decide on the set of arguments to choose, and (2) it is efficient to compute them in linear time as it has been shown in [31]. In the sequel, we do not make any assumption on the chosen semantics, and we assume the user knows how to map the accepted set of arguments to a decision. For example, the reader may refer to [32] to see how argumentation can be used as a decision making process, and to [33] for a survey.

3.3. Argumentation based on logic

Numerous existing works combine logic and argumentation: the works where logics are used to model argumentation [24–27], and the works where argumentation frameworks have arguments structured by a logic [29, 34–38]. The first approach however cannot express arguments based on mental states or normative modalities as the accessibility relation between worlds represents attacks between arguments. That is why we focus here on the second approach.

For instance, [29] proposes to build argumentation frameworks based on propositional logic: an argument is a couple and the attack relation is defined according to logical inconsistencies. Going further, [37] proposes an hypersequential argumentation framework based on a S5 logic. In their approach, $A = (\Gamma, \phi)$ is an argument. Here, Γ is a set of S5-formulas called support and ϕ is the conclusion if, and only if, $\Gamma \Rightarrow \phi$ is a proven sequent. However, [37] does not provide any semantics for their S5-modal operator. Furthermore since they consider only one type

[1] As we want to allow equivalences between some values, we do not consider an irreflexive and asymmetric relation contrary to [22].

of modality, it cannot be used in a general context where several modalities (not necessarily S5 ones) must be considered. Finally, rather than considering a specific logic, [35] proposes a generalization of argumentation based on logic thanks to a function $\mathcal{C}$ representing a Tarskian operator $\vdash$ (i.e. a syntactical proof system).

However all these approaches never provide the formal semantics of the logic they use. Only syntactical proof systems are provided. To the best of our knowledge, while some works deal with semantics of first-order logic [39], there is no argumentation framework based on a normal modal logic semantics. In the sequel, we define such a framework, i.e. an argumentation framework based on a general modal logic semantics.

4. A normal modal logic framework

To represent morals with logics such as DL-MA [8] or dyadic deontic logic [40], we consider a general modal logic where n-ary modal operators [23] are allowed thanks to similarity types. The latter notion represents all modal operators used in the logic and their associated arities. While [23] proposed a semantics for n-ary normal modal logics, they did not propose any general axiomatic system. Hence in this section, we present a n-ary normal modal logic system, that we call the *system GK*, and we give an axiomatization in the next section.

Definition 5 (Similarity type)**.** *A* similarity type *is a couple $\tau = (\mathcal{O}, \rho)$ where $\mathcal{O}$ is a nonempty set of symbols – one for each modal operator – and $\rho : \mathcal{O} \to \mathbb{N}$ assigns an arity to them.*

A similarity type represents a set of modal operators with an assigned arity that is greater than or equal to 0. The elements of $\mathcal{O}$ are usually represented by a 'triangle' like $\triangle_0, \triangle_1, \ldots$ 0-arity modal operators are nullary operators that can be seen as a valuation related to a Kripke frame, and not to a model. As in [8], they can represent for instance choices of agents where a modality $choice(a_i)$ means that in the current world agent i chooses the action a.

Example 2. *In the context of the example, we denote by $\triangle_{car}$, $\triangle_{p.t.}$, $\triangle_{bike}$, $\triangle_{walk}$ the choices of the agent to (reps.) go by car, go by public transport, go by bike or go by foot. They are all nullary modal operators that represent choices.*

If a modal operator $\triangle$ is a monadic operator, i.e. $\rho(\triangle) = 1$, then it is a standard $\Diamond$ modality i.e. an $\exists$-type modality. For $\rho(\triangle) > 1$, a n-ary $\triangle$ means there exists an n-uplet of accessible worlds where each inner formula is true in its respective world of the n-uplet. We denote by ∇ the dual of $\triangle$. When ∇ is a monadic operator, it is equivalent to a standard $\Box$-modality i.e. an $\forall$-type modality. For a similarity type $\tau = (\mathcal{O}, \rho)$, we denote $\mathcal{O}_{dual}$ the set of duals of symbols in $\mathcal{O}$, i.e. all $\forall$-type modalities. For higher arities, n-ary ∇ means that forall n-uplet of accessible worlds, at least one formula is true in its respective world. Interestingly, values can be either represented by nullary operators (expressing values promoted in the current world), either by higher arity operators which can be interpreted as deontic obligations in the name of the values.

Example 3. *For instance we can represent "going by bike promotes the environmental value" either by considering a nullary operator $\triangle_{env}$ that is true in each world where $\triangle_{bike}$ is true, or with a monadic deontic operator from SDL, i.e. $\nabla_{env}(\triangle_{bike})$, or with more expressive n-ary deontic operators. In the sequel, we will consider dyadic modal operators [40] which represent in one parameter the context and in a second parameter what is "ought to be".*

In order to link each modal operator with its dual, we denote $\mathcal{O}^{\Omega} = \mathcal{O} \cup \mathcal{O}_{dual}$ the set of all modal operators and we define a function $\varrho : \mathcal{O}^{\Omega} \to \mathcal{O} \times \mathcal{O}_{dual}$ that gives the mapping between a modality and its dual. In the sequel, we consider that the function that assigns an arity to each operator is extended to this set, i.e. $\rho : \mathcal{O}^{\Omega} \to \mathbb{N}^*$ and obviously the arity function ρ is such that: $\forall \triangle \in \mathcal{O}$, if $\varrho(\triangle) = (\triangle, \nabla)$, then $\rho(\triangle) = \rho(\nabla)$.

Definition 6 (Language of modal logics)**.** *Let $\mathcal{P}$ be a set of propositional atoms and $\tau = (\mathcal{O}, \rho)$ be a similarity type. We define the language of normal modal logic $\mathcal{L}$ i.e. the set of well-formed formulas (wff) by the following BNF grammar, for all $p \in \mathcal{P}$, $\triangle \in \mathcal{O}$:*

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \triangle(\phi_1, \ldots, \phi_{\rho(\triangle)})$$

As usual:

- $\top := \neg\bot$
- $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$
- $\phi \Rightarrow \psi := \neg\phi \vee \psi$

Moreover if $\rho(\triangle) > 0$:

- $\nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) := \neg\triangle(\neg\phi_1, \dots, \neg\phi_{\rho(\triangle)})$

Let us notice that we consider a n-ary K system, as it provides a minimal framework that can be further constrained to express concrete modalities (e.g. seriality for deontic modalities, reflexivility for knowledge, and so on).

Definition 7 (Normal modal logic semantics)**.** *Let $\tau = (\mathcal{O}, \rho)$ be a similarity type. We define a* τ-model *as a tuple $\mathcal{M} = \langle \mathcal{W}, \{\mathcal{R}_\triangle\}_{\triangle\in\mathcal{O}}, \mathcal{V}\rangle$ such that:*

- *$\mathcal{W}$ is a nonempty set of possible worlds,*
- *$\forall\triangle \in \mathcal{O}, \mathcal{R}_\triangle$ is a $(\rho(\triangle)+1)$-ary relation*[2]*,*
- *$\mathcal{V} : \mathcal{P} \to 2^{\mathcal{W}}$ is a valuation function.*

We call $\mathcal{C} = \langle \mathcal{W}, \{\mathcal{R}_\triangle\}_{\triangle\in\mathcal{O}}\rangle$ a τ-frame *and, we say that $\mathcal{M}$ is a model in $\mathcal{C}$ if, and only if, $\mathcal{M}$ belongs to the class of models $\mathcal{C}_{\mathbb{M}} = \{\mathcal{M} : \exists\mathcal{V}, \mathcal{M} = \langle \mathcal{C}, \mathcal{V}\rangle\}$. For all $w \in \mathcal{W}$, and $\phi, \psi \in \mathcal{L}$, $\triangle \in \mathcal{O}$, $\varrho(\triangle) = (\triangle, \nabla)$, and $p \in \mathcal{P}$:*

(1) $\mathcal{M}, w \not\models \bot$
(2) $\mathcal{M}, w \models p$ iff $w \in V(p)$
(3) $\mathcal{M}, w \models \neg\phi$ iff $\mathcal{M}, w \not\models \phi$
(4) $\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$

For the cases where $\rho(\triangle) \geqslant 1$:

(5) $\mathcal{M}, w \models \triangle(\phi_1, \dots, \phi_{\rho(\triangle)})$ if, and only if:

$$\exists(v_1, \dots, v_{\rho(\triangle)}) \in \mathcal{W}^{\rho(\triangle)} : w\mathcal{R}_\triangle(v_1, \dots, v_{\rho(\triangle)}) \text{ and } \forall k \in \mathbb{N}^*, k \leqslant \rho(\triangle) : \mathcal{M}, v_k \models \phi_k$$

For the cases where $\rho(\triangle) = 0$:

(6) $\mathcal{M}, w \models \triangle$ iff $w \in \mathcal{R}_\triangle$

Proposition 1. *Let $w \in \mathcal{W}$. We have $\mathcal{M}, w \models \nabla(\phi_1, \dots, \phi_{\rho(\triangle)})$ if, and only if:*

$$\forall(v_1, \dots, v_{\rho(\triangle)}) \in \mathcal{W}^{\rho(\triangle)} : \text{ if } w\mathcal{R}_\triangle(v_1, \dots, v_{\rho(\triangle)}) \text{ then } \exists k \in \mathbb{N}^*, k \leqslant \rho(\triangle), v_k \models \phi_k$$

Proof. Let $w \in \mathcal{W}$. $\mathcal{M}, w \models \nabla(\phi_1, \dots, \phi_{\rho(\triangle)})$ iff $\mathcal{M}, w \models \neg\triangle(\neg\phi_1, \dots, \neg\phi_{\rho(\triangle)})$ iff $\neg(\exists(v_1, \dots, v_{\rho(\triangle)}) \in \mathcal{W}^{\rho(\triangle)} : w\mathcal{R}_\triangle(v_1, \dots, v_{\rho(\triangle)})$ and $\forall k \in \mathbb{N}^*, k \leqslant \rho(\triangle), v_k \models \neg\phi_k)$ iff $(\forall(v_1, \dots, v_{\rho(\triangle)}) \in \mathcal{W}^{\rho(\triangle)} : \neg w\mathcal{R}_\triangle(v_1, \dots, v_{\rho(\triangle)})$ or $\exists k \in \mathbb{N}^*, k \leqslant \rho(\triangle), v_k \models \phi_k)$ iff $\forall(v_1, \dots, v_{\rho(\triangle)}) \in \mathcal{W}^{\rho(\triangle)}$: if $w\mathcal{R}_\triangle(v_1, \dots, v_{\rho(\triangle)})$ then $\exists k \in \mathbb{N}^*, k \leqslant \rho(\triangle), v_k \models \phi_k$. $\square$

To summarize, in this modal logic framework we have two kinds of modal operators: $\triangle$ is a $(\exists, \wedge)$-type modality represented by the set $\mathcal{O}$, and ∇ is a $(\forall, \vee)$-type modality represented by the set $\mathcal{O}_{dual}$. By $(\exists, \wedge)$-type modality, we mean that it is $\exists$-quantifier on accessible tuplet and a 'conjunction' on the formulas inside the modal operator, while $(\forall, \vee)$-type modality means that it is a $\forall$-quantifier on accessible tuplet and a 'disjunction' on the formulas inside the modal operator. It is important to highlight that for the case where $\rho(\triangle) = 1$ i.e. $\triangle$ is a monadic modal operator, we have $\triangle \equiv \Diamond$ and $\Box \equiv \nabla$. Let us also notice that it would have been possible to provide a semantics for a $(\forall, \wedge)$-type modality and a $(\exists, \vee)$-type modality. However, for the sake of simplicity, we do not consider such types of modality in this article. Let us also notice that, for the same reason, we do not consider the duals of nullary operators.

[2] It is important to notice that each modal operator $\star \in \varrho(\triangle)$ is defined on the same $(\rho(\triangle)+1)$-ary relation.

Example 4. *Let us consider a dyadic deontic formula such as proposed in [40] to express the formula* $O(crisis_{health}, \neg\triangle_{p.t.})$ *meaning that in the case where there is a health crisis, it is forbidden to use public transport. Such deontic operators are used to separate the context from the ideal into two distinct worlds, since representing the context by a naive implication leads standard deontic logic to paradoxes, e.g. Chisholm's Paradox [41]. Thus, in our framework, this dyadic deontic formula can be semantically defined as* $\nabla_O(\neg crisis_{health}, \neg\triangle_{p.t.})$ *which is a* ∇ *operator, thus it is semantically interpreted as a* $(\forall, \vee)$*-modality. Thanks to duality, it can be also semantically defined as* $\neg\triangle_O(crisis_{health}, \triangle_{p.t.})$ *meaning that it is ought not the case that in the context of* $crisis_{health}$ *the formula* $\triangle_{p.t.}$ *is true. More generally, considering three worlds* w_1, w_2, w_3 *such as* $w_1\mathcal{R}(w_2, w_3)$, *the intuitive meaning of the semantics is that in the given world* w_1 *in the context described by the world* w_2 *the ideal world ought to be* w_3.

Let us recall that ϕ is valid in $\mathcal{M}$ (written $\mathcal{M} \models \phi$) if, and only if, for all worlds $w \in \mathcal{W}$, ϕ is satisfied in w i.e. $\mathcal{M}, w \models \phi$ is true. A formula ϕ is valid in a frame $\mathcal{C}$ (written $\models_{\mathcal{C}} \phi$ or $\mathcal{C} \models \phi$) if, and only if, for all models $\mathcal{M}$ built on $\mathcal{C}$, $\mathcal{M} \models \phi$. Now, let $\Gamma \subseteq \mathcal{L}$ be a set of formulas. When Γ is a finite set of formulas, we write $\Gamma \models_{\mathcal{C}} \phi$, meaning ϕ is a *semantic entailment* of Γ if, and only if, for all Kripke models $\mathcal{M}$ in $\mathcal{C}$, if $\mathcal{M} \models \Gamma$ i.e.:

$$\text{If } \mathcal{M} \models \bigwedge_{\psi \in \Gamma} \psi \text{ then } \mathcal{M} \models \phi$$

Let us notice that Γ is written here as a formula because it is finite. For infinite sets of formulas Γ, since well-formed formulas are finite, it should not be allowed to write e.g. $\mathcal{M}, w \models \Gamma$. However, by abuse of notation, and because of the compactness theorems (Theorem 1) hold, when $\mathcal{M}, w \models \Gamma$ is written, it denotes that for all finite subsets $\Gamma_0 \subseteq \Gamma$, $\mathcal{M}, w \models \Gamma_0$ i.e. the conjunction of all formulas in Γ_0 are satisfied in $\mathcal{M}$.

Theorem 1. *Let* Γ *be a set of formulas and* ϕ *be a formula.*
Γ *is satisfiable if, and only if, every finite subsets* $\Gamma_0 \subseteq_f \Gamma$ *are satisfiable.*

The proof is given in Annex A.1.

Example 5. *Let us consider that* $\nabla_{gov}(\phi, \psi)$, *which means that "in the context* ϕ, *it ought to be that* ψ *is true according to the authority", is interpreted as a dyadic* ∇*-modality. Then, we consider the similarity type* $\tau = (\mathcal{O}, \rho)$ *where* $\mathcal{O} = Dyadic_{\triangle} \cup Unary \cup Nullary_{\triangle}$. *Here,* $Dyadic_{\triangle} = \{\triangle_{gov}, \triangle_{env}, \triangle_{sport}, \triangle_{time}, \}$ *denotes all dyadic modal operators that are related respectively to the authority, environmental recommendation, health recommendation to make sport, and what is a saving of time.* $Unary = \{\Diamond\}$ *contains a standard unary possible operator and Nullary* $= \{\triangle_{bike}, \triangle_{p.t.}, \triangle_{walk}, \triangle_{car}\}$. *Thus,* ρ *is such that:* $\forall\star \in Dyadic_{\triangle}, \rho(\star) = 2, \rho(\Diamond) = 1, \forall\star \in Nullary, \rho(\star) = 0$. *Consequently in our running example, we will consider a* τ*-frame* $\mathcal{C} = \langle \mathcal{W}, \{\mathcal{R}_{\triangle}\}_{\triangle \in O} \rangle$. *Let us remark that the unary operator uses neither a* Δ *nor a* ∇ *symbol but rather a* $\Diamond$ *symbol.*

As shown in Example 4, the context can be represented with a negation inside the ∇-modalities. To ease understanding, let us notice that all ∇-type modalities represent a disjunction on the accessible tuple of worlds, e.g. $w \models \nabla(\neg\phi, \psi)$ is true if, and only if, $\forall(u, v) \in \mathcal{R}_{\triangle}(w), u \models \neg\phi$ or $v \models \psi$ i.e. if $u \models \phi$, then $v \models \psi$.

5. Axiomatic system GK

Let us notice that a sound and complete axiomatic system has not been provided for this logic in [23]. Hence, we propose in this article a sound and complete axiomatic system for it. In the sequel, we write $\vdash \phi$ to mean that ϕ is a theorem (i.e. deductible by a Hilbert-proof system). In our axiomatic system we consider the classical axioms of the propositional calculus (PC), the modus ponens (MP), and the uniform substitution (SUB). Moreover, for all $\triangle \in \mathcal{O}$, s.t. $\rho(\triangle) \geqslant 1, \varrho(\triangle) = (\triangle, \nabla)$, we have:

The rule of necessitation, for all $k \in [|1, \rho(\triangle)|]$:

$$\text{From } \vdash \psi_k : \vdash \nabla(\phi_1, \ldots, \phi_{k-1}, \psi_k, \phi_{k+1}, \ldots, \phi_{\rho(\triangle)}) \tag{NEC$^k_\nabla$}$$

The axioms K, for all $k \in [|1, \rho(\triangle)|]$:

$$\vdash \nabla(\phi_1, \ldots, \phi_k \Rightarrow \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)}) \tag{K$^k_\nabla$}$$

The axiom of duality:

$$\vdash \nabla(\phi_1, \ldots, \phi_{\rho(\triangle)}) \Leftrightarrow \neg\triangle(\neg\phi_1, \ldots, \neg\phi_{\rho(\triangle)}) \tag{Dual$_\nabla$}$$

The next rule, called rule of equivalence, is a consequence of axiom K and necessitation.

Theorem 2. *The rule of equivalence is verified, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\textit{From } \vdash \phi_k \Leftrightarrow \psi_k : \vdash \nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)})) \Leftrightarrow \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)}) \tag{RE$^k_\star$}$$

Theorem 3. *The normal properties hold, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\vdash \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Leftrightarrow (\nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)})) \tag{NP^k_∇}$$

$$\vdash \triangle(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow (\triangle(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \triangle(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)})) \tag{$NP^k_\triangle$}$$

Proofs of Theorems 2 and 3 are given in Appendix A.7.

Definition 8 (GK-deductible system)**.** *Let Σ be a set of formulas and ϕ be a formula of $\mathcal{L}$. We say ϕ is* deductible *from Σ, written $\Sigma \vdash \phi$, if, and only if:*

- *if $\Sigma = \emptyset$, then $\vdash \phi$,*
- *else $\exists n \in \mathbb{N}^*, \exists \psi_1, \ldots, \psi_n \in \Sigma, \vdash \psi_1 \wedge \ldots \wedge \psi_n \Rightarrow \phi$.*

Let us notice that, in Definition 8, we allow to write $\Sigma \vdash \phi$ when Σ is infinite. Indeed we have the compactness Theorem 4 w.r.t. GK-deductible systems. Proof of Theorem 4 is given in Appendix A.1.

Theorem 4. *Let Γ be a set of formulas and ϕ be a formula. $\Gamma \vdash \phi$ if, and only if, there exists a finite subsets $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \phi$.*

Let us remark that the notation $\vdash \Gamma \Rightarrow \phi$ corresponds to $\exists \Gamma_0 \subseteq_f \Gamma, \vdash \Gamma_0 \Rightarrow \phi$ (see Theorem 12).

5.1. Soundness

We give below the main theorems needed to prove the soundness of GK. All proofs are given in Appendix A.2.

Theorem 5. *Modus ponens, all tautologies and uniform substitution preserve validity.*

Theorem 6. *The rule of necessitation is valid, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\textit{From } \models \psi_k : \models \nabla(\phi_1, \ldots, \phi_{k-1}, \psi_k, \phi_{k+1}, \ldots, \phi_{\rho(\triangle)}) \tag{NEC$^k_\nabla$}$$

Theorem 7. *The axioms (K^k_∇) is valid, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\models \nabla(\phi_1, \ldots, \phi_k \Rightarrow \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)}) \tag{K$^k_\nabla$}$$

Theorem 8. *The axiom of duality is valid i.e.:*

$$\models \nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) \Leftrightarrow \neg\triangle(\neg\phi_1, \dots, \neg\phi_{\rho(\triangle)}) \tag{Dual$_\nabla$}$$

Theorem 9. *The system GK is sound.*

We can verify that the normal properties are sound in GK. However, they are not needed to prove the soundness.

Theorem 10. *The normal properties are sound, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\models \nabla(\phi_1, \dots, \phi_k \wedge \psi_k, \dots, \phi_{\rho(\triangle)}) \Leftrightarrow (\nabla(\phi_1, \dots, \phi_k, \dots, \phi_{\rho(\triangle)}) \wedge \nabla(\phi_1, \dots, \psi_k, \dots, \phi_{\rho(\triangle)})) \tag{NP_∇}$$

$$\models \triangle(\phi_1, \dots, \phi_k \wedge \psi_k, \dots, \phi_{\rho(\triangle)}) \Rightarrow (\triangle(\phi_1, \dots, \phi_k, \dots, \phi_{\rho(\triangle)}) \wedge \triangle(\phi_1, \dots, \psi_k, \dots, \phi_{\rho(\triangle)})) \tag{$NP_\triangle$}$$

5.2. Completeness

GK is also complete. The detailed proof is given in Appendix A.3.

Theorem 11. *The system GK is complete.*

5.3. Strong soundness and completeness

In this section, we give some important theorems that are verified in GK: the deduction theorems, the strong soundness and the strong completeness. These theorems allows us to obtain the equivalence between the GK-deductibility system and the semantic entailment with respect to all n-ary multi-modal logic frameworks i.e.:

$$\Gamma \vdash \phi \text{ if, and only if, } \Gamma \models \phi$$

The deduction theorems are verified in the logic GK for the finite case but also for the infinite case because we have the compactness theorems (see Theorems 1 and 4). Proofs are given in Appendix A.4.

Theorem 12. *Let Γ be a set of formulas and ϕ be a formula. GK has deduction theorems (1) and (2):*

(1) $\Gamma \vdash \phi$ iff $\vdash \Gamma \Rightarrow \phi$ (Syntactical form)
(2) $\Gamma \models \phi$ iff $\models \Gamma \Rightarrow \phi$ (Semantics form)

The system GK is strongly sound. The proof is given in Appendix A.5.

Theorem 13 (Strong soundness of GK)**.** *Let Γ be a set of formulas of GK, we have that the system GK is strongly sound, i.e. if $\Gamma \vdash \phi$ then $\Gamma \models \phi$.*

The GK system is strongly complete. It means all formulas that are semantic entailments can also be proven by the GK-deductible system. The proof is given in Appendix A.6.

Theorem 14 (Strong completeness of GK)**.** *Let $\phi \in \mathcal{L}$ be a formula. For all sets $\Gamma \subseteq \mathcal{L}$ of formulas, we have that the system GK is strongly complete i.e. if $\Gamma \models \phi$, then $\Gamma \vdash \phi$.*

5.4. *Example of extended GK-system*

As with basic modal logics, we can also add semantic constraints on our n-ary relations. In the running example, we consider one semantic constraint that links the necessity modality with the dyadic deontic operators.

Example 6. *A τ-frame can be constrained to express standard axioms. For instance, the so-called T-axiom can be generalized with a T_τ axiom, i.e. truth axiom for our necessity modality. To this end, we can assume that in our τ-frame, the relation $\mathcal{R}_\triangle$ for the unary $\Diamond$ modality, denoted $\mathcal{R}_\Diamond$, which is assumed to be reflexive and such that:*

$$\forall \triangle' \in Dyadic_\triangle, \forall w \in \mathcal{W}, \forall (v_1, v_2) \in \mathcal{R}_{\triangle'}(w), \exists j \in \{1, 2\}, v_j \in \mathcal{R}_\Diamond(w)$$

Then, one interesting validity we obtain in this τ-frame is:

$$\forall \triangle' \in Dyadic_\triangle : \varrho(\triangle') = (\triangle', \nabla') \quad \models \Box\phi \Rightarrow \nabla'(\phi, \phi)$$

This validity means that if a formula is necessarily true, then it is also true in all accessible worlds. It can be added as a new axiom in the axiomatic system GK, and we can easily show that it preserves strong soundness and strong completeness properties. In the running example, we now assume we have such axiom T_τ for the modal operator $\Box$.

Example 7. *Let us model our transportation system. We want to represent facts (e.g. the agent has a bike, there is a congestion, etc.), contextual obligations in the name of a given value (e.g. it is forbidden to go by public transportation when there is an health crisis in the name of respecting authorities) and choices (e.g. the agent goes by car). Hence, we use nullary modalities to represent choices, monadic modalities to represent facts, i.e. formulas that are necessary, and dyadic modalities to represent necessary formulas in a given context from a given value point-of-view. Let us then consider the following hypothesis:*

$$\Gamma_1 \stackrel{\text{def}}{=} \{\nabla_{gov}(\neg crisis_{health}, \neg\triangle_{p.t.})\}$$

$$\Gamma_2 \stackrel{\text{def}}{=} \{\triangle_{car} \rightarrow has_{car}, \triangle_{bike} \rightarrow has_{bike}\}$$

$$\Gamma_3 \stackrel{\text{def}}{=} \{\nabla_{sport}(\bot, \triangle_{bike})\}$$

$$\Gamma_4 \stackrel{\text{def}}{=} \{\nabla_{time}(\neg emergency \vee congestion, \triangle_{car})\}$$

$$\Gamma_5 \stackrel{\text{def}}{=} \{\neg\triangle_{time}(emergency, \triangle_{walk})\}$$

$$\Gamma_6 \stackrel{\text{def}}{=} \{\nabla_{env}(\bot, \triangle_{bike} \vee \triangle_{walk} \vee \triangle_{p.t.})\}$$

$$\Gamma_7 \stackrel{\text{def}}{=} \{\nabla_{env}(\neg congestion, \neg\triangle_{car})\}$$

$$\Gamma_8 \stackrel{\text{def}}{=} \{\Box \texttt{XOR}(\triangle_{car}, \triangle_{p.t.}, \triangle_{bike}, \triangle_{walk})\}$$

$$\Gamma_9 \stackrel{\text{def}}{=} \{\Box(has_{car} \wedge \neg has_{bike} \wedge crisis_{health} \wedge \neg congestion \wedge \neg emergency)\}$$

Γ_1 means that the authorities recommend to not take public transport when there is a health crisis. Γ_2 represents an obvious condition: biking or driving implies that the agent has a bike or a car. Γ_3 means, whatever the context is, cycling allows you to exercise. Γ_4 means if there is an emergency and there is no congestion, then it saves time to take car. Γ_5 means it is not possible that in an emergency, it saves time to walk. Γ_6 means, whatever the context is, it is good for environment to take bike, walk or take public transport. Γ_7 means it is good for environment to not take the car when there is a congestion. Γ_8 means it is necessary that the agent chooses only one choice at the same time (e.g. she cannot take the car and the bike simultaneously). Γ_9 describes the initial situation of the agent.

Example 8. *From Example 7, we can easily show:*

$$\Gamma_5 \vdash \nabla_{time}(\neg emergency, \neg\triangle_{walk})$$

$$\Gamma_2 \vdash \neg has_{bike} \Rightarrow \neg\triangle_{bike}$$

$$\Gamma_2 \cup \Gamma_9 \vdash \neg\triangle_{bike}$$

$$\Gamma_6 \cup \Gamma_8 \vdash \nabla_{env}(\bot, \neg\triangle_{car})$$

$$\Gamma_8 \vdash \texttt{XOR}(\triangle_{car}, \triangle_{p.t.}, \triangle_{bike}, \triangle_{walk})$$

$$\Gamma_9 \vdash \neg emergency \vee congestion$$

Let us notice we deduce from Γ_6 and Γ_8 that, whatever the context is, it is not good for environment to take the car, which is stronger than the assumption Γ_7.

6. Modal logic-based VAF

To define a VAF based on modal logic, we first define arguments, and then attacks between arguments when they are structured on a generalized modal logic.

6.1. Arguments

In the literature, a logic-based argument is a couple (Γ, ϕ) where Γ is a set of formulas, called *premises* or *support*, that implies a formula ϕ, called the *conclusion* [29, 35–38]. Usually, such arguments are defined based on its syntactical implication, implicitly assuming that the underlying logic is sound and complete. Since our aim is to define a modal logic for abstract argumentation that could express arguments as mental states, norms, laws, etc., we first adopt a semantically-oriented definition for valid arguments, and then we adopt and use a rigorous definition for deductive arguments (based on syntactical proof systems). Hence, we call a *valid argument* a couple (Γ, ϕ) where the conclusion ϕ is a semantic consequence of the set of premises Γ. This definition is an adaptation of [29] with respect to the modal logic semantics.

Definition 9 (Valid argument)**.** *Let $\tau = (\mathcal{O}, \rho)$ be similarity type, and $\mathcal{C}$ be a τ-frame. A* valid argument *in $\mathcal{C}$ is a couple $A = (\Phi, \alpha)$ s.t.:*

(1) $\Phi \subseteq \mathcal{L}$ is a finite subset of $\mathcal{L}$
(2) $\Phi \not\models_{\mathcal{C}} \bot$ i.e. Φ is not inconsistent: $\exists \mathcal{M}, w \models \Phi$
(3) $\Phi \models_{\mathcal{C}} \alpha$
(4) Φ is a minimal set w.r.t. $\subseteq$ s.t. (1), (2) and (3) are respected.

Here Φ is called the ***support*** *or the set of* ***premises*** *of argument A, denoted $P(A) = \Phi$, and α its* ***conclusion****, denoted $C(A) = \alpha$.*

When an argument is called a *deductive argument* it is defined with a syntactical proof system i.e. with the Tarskian operator $\vdash$ as in sequent-based argumentation [36]. Obviously, we have the equivalence between deductive arguments and valid arguments in sound and complete frameworks. For convenience of notation, we will write $A = (\_, \phi)$ as a shortcut meaning the set of premises Γ is not specified as it is not needed for our explanation (this is a standard notation in semantic web, called a *blank node*). In a rigorous manner, blank nodes _ denote here a $\exists \Gamma \subseteq \mathcal{L}$ s.t. Γ is finite and $A = (\Gamma, \phi)$ is a valid argument.

Example 9. *Some examples of deductive arguments:*

$$A_1 = (\Gamma_1, \neg congestion)$$

$$A_2 = (\Gamma_7, \nabla_{env}(\neg congestion, \neg\triangle_{car}))$$

$$A_3 = (\Gamma_9, \Box\neg congestion)$$

$$A_4 = (\Gamma_8, \Box \texttt{XOR}(\triangle_{car}, \triangle_{p.t.}, \triangle_{bike}, \triangle_{walk}))$$

$$A_5 = (\Gamma_3, \nabla_{sport}(\bot, \triangle_{bike}))$$

$$A_6 = (\Gamma_2 \cup \Gamma_9, \Box\neg\triangle_{bike})$$

$$A_7 = (\Gamma_4, \nabla_{time}(\neg emergency \vee congestion, \triangle_{car}))$$

$$A_8 = (\Gamma_9, \neg emergency \vee congestion)$$

$$A_9 = (\Gamma_5, \nabla_{time}(\neg emergency, \neg\triangle_{walk}))$$

$$A_{10} = (\Gamma_9, \neg emergency)$$

$$A_{11} = (\Gamma_6 \cup \Gamma_8, \nabla_{env}(\bot, \neg\triangle_{car}))$$

$$A_{12} = (\Gamma_1, \nabla_{gov}(\neg crisis_{health}, \neg\triangle_{p.t.}))$$

Obviously, other arguments can be deduced, e.g. $(\Gamma_9, crisis_{health})$. *But since their conclusion does not contain a decision (i.e. a nullary modality) or they do not attack another argument, we do not represent them.*

6.2. Modal attacks

Several kinds of attack relations have been defined for classical logic-based argumentation [29, 34, 42, 43]. We recall the semantics of the best known, namely rebuttal, undercut, and defeater. The interested reader may refer to [34] for other specific attacks such as direct defeater or canonical undercut.

Definition 10 (Rebuttal)**.** *We say that* $A = (\Phi, \alpha)$ *is a* rebuttal *for* $B = (\Psi, \beta)$ *– written A Reb B – if, and only if,* $\models \alpha \Leftrightarrow \neg\beta$ *(i.e.* $\alpha \Leftrightarrow \neg\beta$ *is a tautology).*

Definition 11 (Undercut attack)**.** *We say that* $A = (\Phi, \alpha)$ *is an* undercut *for* $B = (\Psi, \beta)$ *– written A Und B – if, and only if, there exists:*

$$\Psi' = \{\psi_1, \dots, \psi_n\} \subseteq \Psi \text{ such that } \models \alpha \Leftrightarrow \neg \bigwedge_{\psi_i \in \Psi'} \psi_i$$

Definition 12 (Defeater attack)**.** *We say that* $A = (\Phi, \alpha)$ *is a* defeater *for* $B = (\Psi, \beta)$ *– written A Def B – if, and only if, there exists:*

$$\Psi' = \{\psi_1, \dots, \psi_n\} \subseteq \Psi \text{ such that } \alpha \models \neg \bigwedge_{\psi_i \in \Psi'} \psi_i$$

However these attacks cannot express attacks between different modal operators since there is no necessary inconsistency.

Example 10. *From the previous example, let us consider both arguments* A_7 *and* A_{11} *that conclude (resp.) "if there is an emergency and there is no congestion, then it saves time to go by car" and "whatever the context is, it is good for the environment to not go by car". Since* ∇_{time} *and* ∇_{env} *are two different modal operators that describe ideals and do not imply that both inner formula are true in the same worlds, there is no inconsistency. Hence, the standard attacks given in Definitions 10, 11 and 12 cannot catch the intuitive attack between* A_7 *and* A_{11}.

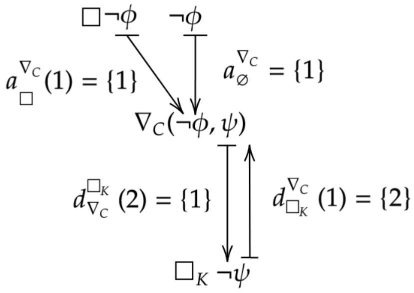

Fig. 1. Modal mapping between facts, $\Box$, ∇_C and $\Box_K$

Thus we propose a new kind of attack able to deal with any normal modal logic, called a *modal attack*. As for the standard attacks, we have a notion of *modal rebuttal*, *modal undercut* and *modal defeater*. Defining attacks with a general modal logic framework is non-trivial since modal operators can be associated with different arities.

Let us consider a dyadic deontic formula $\nabla_C(\neg\phi, \psi)$ that means in the case where ϕ is true, it is obligatory to have ψ true, and a monadic deontic formula $\Box_K\neg\psi$. Let us assume that ψ denotes "say a lie" and ϕ denotes "a life is threatened". According to $\Box_K$ it is not allowed to lie regardless of the context, while for ∇_C it would be necessary to lie in the context ϕ. Intuitively, an attack between two arguments $C = (\_, \nabla_C(\neg\phi, \psi))$ and $K = (\_, \Box_K\neg\psi)$ would rely on the inconsistency between the second parameter of the dyadic operator ∇_C and the monadic operator $\Box_K$. However, C should be defeated if ϕ is false, i.e. the context is not verified. Thus, an argument $F = (\_, \neg\phi)$ that concludes "it is a fact that no life is threatened" should defeat C.

On the first hand, since the modal operator $\Box_K$ is assumed to be a monadic operator and ∇_C is a dyadic operator, the attack between K and C is grounded on a contradiction between the first parameter of $\Box_K$ and the second parameter of ∇_C. We say those parameters are *disaligned*. On the other hand, the attack between F and C is grounded on an equivalence between $\neg\phi$ and the first parameter of ∇_C. We say those parameters are *aligned*. Thus, we need to specify which parameters attacks is grounded on, and how to deal with the nature of the attacks: *aligned* or *disaligned*.

Figure 1 illustrates how we characterize such relationships between formulas. Firstly, we define a $d_{\star_1}^{\star_2}$*-mapping* which maps the parameters of two distinct modalities $\star_1$ and $\star_2$, that must be inconsistent to characterize an attack. In the previous example, $K = (\_, \Box_K\psi_1)$ with $\psi_1 := \neg\psi$, and $C = (\_, \nabla_C(\phi_1, \phi_2))$ with $\phi_1 := \neg\phi$ and $\phi_2 := \psi$, mutually attack each other as $\models \phi_2 \equiv \neg(\psi_1)$. Thus, the d-mapping is defined as $d_{\Box_K}^{\nabla_C} = \{(1, 2)\}$ i.e. the first parameter of $\Box_K$ is associated with the second parameter of ∇_C and they have to be in contradiction to define a modal attack.

Secondly, we define a $a_{\star_1}^{\star_2}$*-mapping* which maps the parameters of two distinct modalities that must be equivalent to characterize an attack. Let us consider the standard modality $\Box$ and the formula $\Box\theta_1$, where $\theta_1 := \neg\phi$, meaning "$\neg\phi$ is necessary true". Let us consider $\nabla_C(\phi_1, \phi_2)$. As θ_1 is necessary true i.e. $\neg\phi$, it cannot be possible that the context of $\nabla_C(\phi_1, \phi_2)$ is verified, i.e. ϕ is true. Thus, the a-mapping is defined as $a_{\Box}^{\nabla_C} = \{(1, 1)\}$ and is logically represented by the validity $\models \phi_1 \equiv \theta_1$. While it can be counter-intuitive to consider equivalence to characterize attacks, it makes sense for the modality ∇ since the contextual parameter is represented by the negation.

Thirdly, we define a $d_{\emptyset}^{\star_2}$*-mapping* (resp. $a_{\emptyset}^{\star_2}$*-mapping*) which characterizes the parameters of $\star_2$ that must be inconsistent (reps. equivalent) to a factual truth, i.e. an argument whose conclusion is not preceded by a modal operator. For instance, $a_{\emptyset}^{\nabla_C} = \{1\}$ means the contextual parameter of the dyadic deontic operator ∇_C can be attacked by factual truths.

Instead of explicitly denoting four distinct mappings, we propose a compact notation where both d- and a-mappings are jointly represented as a set of sets of rules, where rules describe which parameters attack which parameters for a given kind of mapping. For instance, we can represent a $\langle d, a\rangle_{\star_1}^{\star_2}$-mapping such as $\{\{(1, 2, d), (2, 3, a)\}, \{(3, 4, d)\}\}$. Here, $\langle d, a\rangle_{\star_1}^{\star_2}$ can be viewed as a CNF formula of rules that must be satisfied to have an attack, i.e. there is an attack if (parameter 1 in $\star_1$ and parameter 2 in $\star_2$ are inconsistent, or parameter 2 in $\star_1$ and parameter 3 in $\star_2$ are equivalent) and parameter 3 in $\star_1$ and parameter 4 in $\star_2$ are inconsistent.

Definition 13 ($\langle d,a\rangle$-modal mappings)**.** *Let $\tau = (\mathcal{O},\rho)$ be a similarity type, $\mathcal{O}^\Omega$ be the set of modal symbols based on τ, $\{d,a\}$ be a set of symbols (that corresponds to the types of mappings we want to define), and $(\star_1,\star_2) \in \mathcal{O}^\Omega \times \mathcal{O}^\Omega$ be two modalities.*

We say that $\langle d,a\rangle^{\star_2}_{\star_1}$ is a modal mapping *for $(\star_1,\star_2)$ iff:*

$$\langle d,a\rangle^{\star_2}_{\star_1} \subseteq 2^{[|1,\rho(\star_1)|]\times[|1,\rho(\star_2)|]\times\{d,a\}}$$

We say that $\langle d,a\rangle^{\star_2}_{\emptyset}$ is a modal mapping *for $\star_2$ iff:*

$$\langle d,a\rangle^{\star_2}_{\emptyset} \subseteq 2^{[|1,\rho(\star_2)|]\times\{d,a\}}$$

We call $\langle d,a\rangle_\tau$ a modal mapping for the τ-frame*, a function that associates for each pair of modalities $(\star_1,\star_2) \in \mathcal{O}^\Omega \times \mathcal{O}^\Omega$ a modal mapping for $(\star_1,\star_2)$ and for each modality $\star_2 \in \mathcal{O}^\Omega$ assigns a modal mapping for $\star_2$, i.e. $\langle d,a\rangle_\tau$ is such that:*

- $\langle d,a\rangle_\tau : (\mathcal{O}^\Omega \cup \{\emptyset\}) \times \mathcal{O}^\Omega \to 2^{2^{((\mathbb{N}^*\times\mathbb{N}^*)\cup\mathbb{N}^*)\times\{d,a\}}}$
- *For all $(\star_1,\star_2) \in \mathcal{O}^\Omega \times \mathcal{O}^\Omega$, $\langle d,a\rangle_\tau(\star_1,\star_2) \subseteq 2^{[|1,\rho(\star_1)|]\times[|1,\rho(\star_2)|]\times\{d,a\}}$*
- *For all $\star_2 \in \mathcal{O}^\Omega$, $\langle d,a\rangle_\tau(\emptyset,\star_2) \subseteq 2^{[|1,\rho(\star_2)|]\times\{d,a\}}$*

In the sequel, we consider *consistent* modal mappings. Firstly, it means that a modal mapping must be possible in practice, i.e. can be satisfiable, since a modal mapping for two modalities can be viewed as a CNF formula. Indeed, if for two modalities we have $\langle d,a\rangle^{\star_2}_{\star_1} = \{\{(i,j,d)\},\{(i,j,a)\}\}$, then it would characterize parameters that should be both inconsistent and equivalent, which makes no sense. Secondly, when a modal mapping has not been explicitly defined, we assume the empty set for each pair of modalities, or for a single modality.

Definition 14 (Consistent modal mappings)**.** *Let $\tau = (\mathcal{O},\rho)$ be a similarity type, $\mathcal{O}^\Omega$ be the set of modal symbols based on τ, $\langle d,a\rangle_\tau$ be a modal mapping for the τ-frame, and $\mathcal{P} = \{p_i,q_j : (i,j) \in [|1,\rho(\star_1)|] \times [|1,\rho(\star_2)|], (\star_1,\star_2) \in (\mathcal{O}^\Omega)^2\} \cup \{q_j : j \in [|1,\rho(\star_2)|], \star_2 \in \mathcal{O}^\Omega\}$ be a set of propositional atoms. We say that $\langle d,a\rangle_\tau$ is a* consistent modal mapping *iff the following propositional formula ϕ is satisfiable:*

$$\phi = \bigwedge_{(\star_1,\star_2)\in\mathcal{O}^\Omega\times\mathcal{O}^\Omega} \Big(\bigwedge_{S\in\langle d,a\rangle^{\star_2}_{\star_1}} \Big(\bigvee_{\substack{(i,j,t)\in S:\\ t=a}} (p_i \Leftrightarrow q_j) \vee \bigvee_{\substack{(i,j,t)\in S:\\ t=d}} (p_i \Leftrightarrow \neg q_j)\Big)\Big) \wedge \bigwedge_{\star_2\in\mathcal{O}^\Omega} \Big(\bigwedge_{S\in\langle d,a\rangle^{\star_2}_{\emptyset}} \Big(\bigvee_{\substack{(j,t)\in S:\\ t=a}} q_j \vee \bigvee_{\substack{(j,t)\in S:\\ t=d}} \neg q_j\Big)\Big)$$

It is a direct translation from each modal mapping $(i,j)^{\star_2}_{\star_1}$ to a propositional atom p_i that represents the formula at the i-th parameter of $\star_1$ and q_j represents the formula at the j-th parameter of $\star_2$. When it is aligned, we translate it as $p_i \Leftrightarrow q_j$ and use the negation as $p_i \Leftrightarrow \neg q_j$ when it is disaligned. Let us notice that we could have written $\neg p_i \Leftrightarrow q_j$ which is logically equivalent. We follow the same idea for $(j)^{\star_2}_{\emptyset}$. However only q_j are involved in the formula since there is only a need of mapping the parameters of $\star_2$. Finally, verifying if a given modal mapping is a consistent one is NP-complete since it is equivalent to solve the SAT problem for propositional logics.

Example 11. *In our running example, we assume a disaligned modal mapping between each dyadic deontic operator on its second parameter which represents the "ought to be" part. While we consider an aligned modal mapping between the first parameter of the dyadic operators, and factual verities (or necessary verities). Thus, the $\langle d,a\rangle_\tau$ modal mapping is such that:*

- $\forall\star_1,\star_2 \in \mathit{Dyadic}_\triangle, \langle d,a\rangle_\tau(\star_1,\star_2) = \{\{(2,2,d)\}\}$
- $\forall\star_2 \in \mathit{Dyadic}_\triangle, \langle d,a\rangle_\tau(\emptyset,\star_2) = \{\{(1,a)\}\}$
- $\forall\star_2 \in \mathit{Dyadic}_\triangle, \langle d,a\rangle_\tau(\Box,\star_2) = \{\{(1,1,a)\}\}$

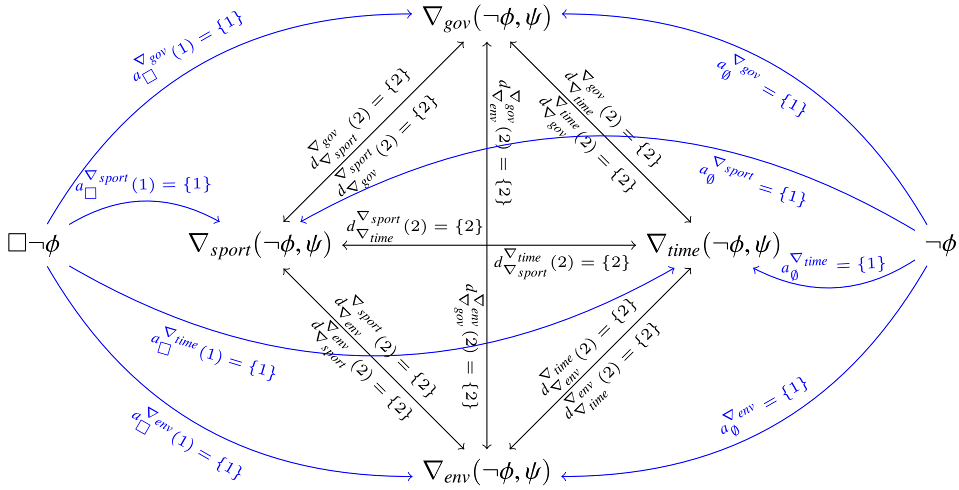

Fig. 2. Modal mapping of Example 11

Figure 2 shows the complete modal mapping. Blue relations represent the modal mappings between facts, necessity and dyadic deontic modal operators, while black relations represent modal mappings between all dyadic deontic modal operators. For instance, let us consider the black arrow from formula $\nabla_{env}(\neg\phi,\psi)$ to formula $\nabla_{time}(\neg\phi,\psi)$. The modal mapping $d_{\nabla_{env}}^{\nabla_{time}}(2) = \{2\}$ means the second parameter of both modalities must be disaligned in order to express a contradiction. Let us notice that this notation is equivalent to write $d_{\nabla_{env}}^{\nabla_{time}} = \{(2,\{2\})\}$. By the same way, we write for the corresponding complete modal mapping $\langle d,a\rangle_\tau(\nabla_{env},\nabla_{time}) = \{\{(2,2,d)\}\}$.

These modal mappings now make it possible to define several kinds of modal logical attacks such as modal rebuttals, modal undercuts, or modal defeaters. However, due to the complex interactions between modalities and mapping, the number of attacks to characterize is huge. To factorize notations, we introduce a function Υ that expresses compact attack rules that will be used to characterize modal rebuttal, modal undercut, and modal defeater. The idea is to split the definition in two cases: rules that represent attacks between two modalities, and rules that represent attacks between facts and modalities. Furthermore to keep the definition compact, we introduce a parameter in Υ, denoted by *type*, to make distinction between rebuttals / undercuts ($\Leftrightarrow$), and defeaters ($\Rightarrow$).

Definition 15 (Υ-attack rule for $\langle d,a\rangle$-mapping)**.** *Let $\tau = (\mathcal{O},\rho)$ be a similarity type, $\mathcal{O}^\Omega$ be the set of modal symbols based on τ, $\langle d,a\rangle$ be a modal mapping for the τ-frame, and $\Upsilon = \{\Upsilon_\star^{\star_2}\}_{(\star,\star_2)\in M}$ be a set of functions where $M = (\mathcal{O}^\Omega \cup \{\emptyset\}) \times \mathcal{O}^\Omega$:*

$$\forall(\star,\star_2) \in M, \Upsilon_\star^{\star_2} : \mathcal{L} \times \mathcal{L} \times \{\Rightarrow,\Leftrightarrow\} \to \{\bot,\top\}$$

For all $(\phi,\psi) \in \mathcal{L}^2$, for all $(\star_1,\star_2) \in \mathcal{O}^\Omega \times \mathcal{O}^\Omega$. We call Υ the set of attack rules for the $\langle d,a\rangle$-mapping *if Υ is such that (1) and (2) hold:*

(1) If $\phi = \star_1(\phi_1,\ldots,\phi_{\rho(\star_1)})$, for all $i \in [|1,\rho(\star_1)|]$, $\phi_i \equiv \phi_i^1 \wedge \ldots \wedge \phi_i^{n_i}$ be its equivalent canonical Conjunctive Normal Form (CNF), $\psi = \star_2(\psi_1,\ldots,\psi_{\rho(\star_2)})$, $\Upsilon_{\star_1}^{\star_2}(\phi,\psi,type) = \top$ if, and only if, $\forall S \in \langle d,a\rangle_{\star_1}^{\star_2}$, $\exists(i,j,t) \in S$, $\exists K \subseteq [|1,n_i|]$:

- *if $t = d$, then:*

$$\{\bigwedge_{k\in K}\phi_i^k\} \models \neg\psi_j \text{ and } (\text{if } type = \Leftrightarrow, \{\neg\psi_j\} \models \bigwedge_{k\in K}\phi_i^k)$$

- *if $t = a$, then:*

$$\{\bigwedge_{k\in K}\phi_i^k\} \models \psi_j \text{ and } (\text{if } type = \Leftrightarrow, \{\neg\psi_j\} \models \bigwedge_{k\in K}\phi_i^k)$$

(2) If $\phi \equiv \Phi$ where $\Phi = \phi^1 \wedge \ldots \wedge \phi^n$ be its equivalent canonical CNF of ϕ, and $\psi = \star_2(\psi_1, \ldots, \psi_{\rho(\star_2)})$, then $\Upsilon^{\star_2}_{\emptyset}(\phi, \psi, type) = \top$ if, and only if, $\forall S \in \langle d,a\rangle^{\star_2}_{\emptyset}$, $\exists (j,t) \in S, \exists K \subseteq [|1,n|]$:

- *if $t = d$, then:*

$$\{\bigwedge_{k\in K} \phi^k\} \models \neg\psi_j \text{ and } (\text{if } type = \Leftrightarrow, \{\neg\psi_j\} \models \bigwedge_{k\in K} \phi^k)$$

- *if $t = a$, then:*

$$\{\bigwedge_{k\in K} \phi^k\} \models \psi_j \text{ and } (\text{if } type = \Leftrightarrow, \{\psi_j\} \models \bigwedge_{k\in K} \phi^k)$$

The notion of attack rules is illustrated in Example 12.

Example 12. *Let consider $A_7 = (\Gamma_4,\ \nabla_{time}(\neg emergency \vee congestion, \triangle_{car}))$ and $A_{11} = (\Gamma_6 \cup \Gamma_8, \nabla_{env}(\bot, \neg\triangle_{car}))$. Here, we notice we would like to draw an attack from the contradiction between the "ideal" parameters (i.e. the second parameter) in ∇_{time} and ∇_{env}. The first argument concludes that in the situation where $\neg emergency \vee congestion$ is not verified, i.e. there is an emergency and no congestion, going by car promotes the time value. The second argument concludes that going by car never promotes sustainability. However in point of view of the modal logic, there is no contradiction since the truth value is contained inside distinct modalities. To map the contradiction between those parameters, we defined in Example 11 the mapping $\langle d,a\rangle_\tau(\nabla_{time}, \nabla_{env}) = \langle d,a\rangle_\tau(\nabla_{env}, \nabla_{time}) = \{\{(2,2,d)\}\}$ which means that the parameters must be disaligned. Now, we need a function, called $\Upsilon^{\nabla_{env}}_{\nabla_{time}}$, to characterize an attack based on a $\langle d,a\rangle$-mapping. Let us consider $\phi_2 = \triangle_{car} = \phi_2^1$ and $\psi_2 = \neg\triangle_{car} = \psi_2^1$ the formulas for the second parameter of ∇_{time} and ∇_{env} respectively. Obviously, since $\forall S \in \langle d,a\rangle^{\nabla_{env}}_{\nabla_{time}}$, $(2,2,d) \in S$, $\{1\} \subseteq [|1,n_i|]$ we have $\{\phi_2^1\} \models \neg\psi_2^1$, we deduce that $\Upsilon^{\nabla_{env}}_{\nabla_{time}}(\phi_1, \psi_2, \Rightarrow) = \top$. Furthermore, for the same reasons and $\{\psi_2^1\} \models \neg\phi_2^1$, we also deduce $\Upsilon^{\nabla_{env}}_{\nabla_{time}}(\phi_1, \psi_2, \Leftrightarrow) = \top$. Thus we can conclude that there is a "modal-rebuttal attack" when $\Upsilon^{\nabla_{env}}_{\nabla_{time}}(\phi_1, \psi_2, \Leftrightarrow) = \top$.*

We then define the standard attacks in logic-based argumentation.

Definition 16 (Types of modal attack)**.** *Let $\tau = (\mathcal{O}, \rho)$ be a similarity type, and $\langle d,a\rangle$ be a consistent modal mapping for the τ-frame, and $(\star, \star_2) \in (\mathcal{O}^\Omega \cup \{\emptyset\}) \times \mathcal{O}^\Omega$,*

- *An argument A is a* direct $\langle d,a\rangle^{\star_2}_{\star}$modal rebuttal *for B, written $(A,B) \in$ Reb $\langle d,a\rangle^{\star_2}_{\star}$ if, and only if, $A = (\_, \phi)$, $B = (\_, \star_2(\psi_1, \ldots, \psi_{\rho(\star_2)}))$ and*

$$\Upsilon^{\star_2}_{\star}(\phi, \star_2(\psi_1, \ldots, \psi_{\rho(\star_2)}), \Leftrightarrow) = \top$$

- *An argument A is a* direct $\langle d,a\rangle^{\star_2}_{\star}$modal undercut *for B, written $(A,B) \in$ Und $\langle d,a\rangle^{\star_2}_{\star}$ if, and only if, $A = (\_, \phi)$, $B = (\Psi, \_)$, there exists $\psi \in \Psi$, $\psi = \star_2(\psi_1, \ldots, \psi_{\rho(\star_2)})$,*

$$\Upsilon^{\star_2}_{\star}(\phi, \star_2(\psi_1, \ldots, \psi_{\rho(\star_2)}), \Leftrightarrow) = \top$$

- *An argument A is a* direct $\langle d,a\rangle^{\star_2}_{\star}$modal defeater *for B, written $(A,B) \in$ Def $\langle d,a\rangle^{\star_2}_{\star}$ if, and only if, $A = (\_, \phi)$, $B = (\Psi, \_)$, there exists $\psi \in \Psi$, $\psi = \star_2(\psi_1, \ldots, \psi_{\rho(\star_2)})$,*

$$\Upsilon^{\star_2}_{\star}(\phi, \star_2(\psi_1, \ldots, \psi_{\rho(\star_2)}), \Rightarrow) = \top$$

- *We define a $\langle d,a\rangle^{\star_2}_{\star}$*modal attack *as:*

$$Att\langle d,a\rangle^{\star_2}_{\star} = Reb\ \langle d,a\rangle^{\star_2}_{\star} \cup Und\ \langle d,a\rangle^{\star_2}_{\star} \cup Def\ \langle d,a\rangle^{\star_2}_{\star}$$

Notice that since nullary modal operators are true in worlds for all models of the frame, they are similar to facts. Indeed, the nullary operators generally and intuitively represent choices as e.g. "the agent goes by bike" and "the agent goes by car". Thus, both choices cannot be true in the same world. Indeed if they were, then it would say "the agent goes by bike and car at the same time", which makes no sense in this case. Hence, the attacks between nullary operator can be defined as logical contradictions and thus we do not need to define modal mappings for them.

6.3. Modal mapping properties

Due to the structured nature of arguments, classical attacks are either symmetric, or involved in a circular relationship, which can be problematic. Indeed, it is problematic for two main reasons: (1) symmetry can lead to exhibit many distinct admissible sets of arguments, which can in turn lead to a dilemma i.e. how to chose the set of arguments to justify a choice; (2) interesting semantics due to their uniqueness, such as the grounded semantics, cannot be considered.

Lemma 1. *Let $A = (\Gamma, \phi)$, $B = (\Sigma, \psi)$ be two arguments,*

(1) If A Reb B, then B Reb A,
(2) If A Und B, then there exists an argument C such that C Def A,
(3) If A Def B, then there exists an argument C such that C Def A.

Proof. Let $A = (\Gamma, \phi)$ and $B = (\Sigma, \psi)$ be two arguments.

(1) If A Reb B, then it is obvious by $\models \neg\phi \equiv \psi$.
(2) If A Und B, then $\Gamma \models \phi, \Sigma \models \psi$ and there exists $\Sigma' \subseteq \Sigma$, $\models \phi \equiv \neg \bigwedge_{\psi_i \in \Sigma'} \psi_i$. We denote $\theta = \bigwedge_{\psi_i \in \Sigma'} \psi_i$. Thus, $\models \neg\phi \equiv \theta$. But since $A = (\Gamma, \phi)$ is an argument, $\Gamma \models \phi$ and $\models (\Gamma \Rightarrow \phi) \Rightarrow \neg\phi \rightarrow \neg\Gamma$, thus $\{\neg\phi\} \models \neg\Gamma$ by the deduction theorem. Since $\models \theta \equiv \neg\phi$ and $\{\neg\phi\} \models \neg\Gamma$, we have $\{\theta\} \models \neg\Gamma$. Consequently, (Σ, θ) Def (Γ, ϕ) and so there exists $C = (\Sigma, \theta)$, C Def A.
(3) If A Def B, then $\Gamma \models \phi, \Sigma \models \psi$ and there exists $\Sigma' \subseteq \Sigma$, $\{\phi\} \models \neg \bigwedge_{\psi_i \in \Sigma'} \psi_i$. We denote $\theta = \bigwedge_{\psi_i \in \Sigma'} \psi_i$. Since $\models (\phi \Rightarrow \neg\theta) \Rightarrow (\theta \Rightarrow \neg\phi)$, we deduce by the deduction theorem $\{\theta\} \models \neg\phi$. Since $\Gamma \models \phi$ iff $\{\neg\phi\} \models \neg\Gamma$, we have $\{\theta\} \models \neg\Gamma$. Consequently, (Σ, θ) Def (Γ, ϕ) and so there exists $C = (\Sigma, \theta)$, C Def A.

□

Hence, the properties of standard logic-based attacks are what we call *quasi-symmetric*, i.e. when an argument A attacks another, A is necessarily attacked in return. However interestingly, our modal attacks do not have this property.

Definition 17. *A binary relation $\mathcal{R}$ is quasi-symmetric iff $\forall (a,b) \in \mathcal{R}, \exists c : (c,a) \in \mathcal{R}$.*

Theorem 15. *The relation Reb $\cup$ Und $\cup$ Def is quasi-symmetric while $Att\langle d,a\rangle^{\star 2}_{\star}$ is not quasi-symmetric.*

Proof. By using the Lemma 1, it is trivial for classical attacks. Figure 3 provides a counter-example, showing $Att\langle d,a\rangle^{\star 2}_{\star}$ is not quasi-symmetric. □

Finally, let us notice we respect Amgoud's postulates [35] which are desirable properties expected from any well-defined logic-based argumentation system. Let Γ be a finite set of formulas which represents a knowledge base.

- We define $\mathcal{C}$ as:
$$C(\Gamma) = \{\phi : \Gamma \models \phi\}$$
- Obviously we have $C(\Gamma) = C(C(\Gamma))$ i.e. idempotence.
- If Γ is not consistent, then $C(\Gamma) = \mathcal{L}$. Obvious since Γ is not consistent iff $\Gamma \models \bot$. So for all formulas $\phi \in \mathcal{L}$, since $\phi \wedge \neg\phi \equiv \bot$, we have $\Gamma \models \phi \wedge \neg\phi$, and $\models \phi \wedge \neg\phi \Rightarrow \phi$, then $\Gamma \models \phi$, thus for all formulas $\phi \in \mathcal{L}$, $\phi \in C(\Gamma)$ i.e. $C(\Gamma) = \mathcal{L}$.
- If Γ is consistent, then $C(\Gamma) \neq \mathcal{L}$. By absurd, let assume that $C(\Gamma) = \mathcal{L}$, since $\phi \wedge \neg\phi \in \mathcal{L}$, we have $\phi \wedge \neg\phi \in C(\Gamma)$. Thus by definition, $\Gamma \models \phi \wedge \neg\phi$. Thus by deduction theorem, $\models \Gamma \Rightarrow \bot$ which is absurd w.r.t. that Γ was consistent. Consequently, Γ is consistent iff C is a consistent function.
- Absurdity's postulate holds too i.e. there exists $\phi \in \mathcal{L}$ (e.g. $\phi = \bot$) s.t. $C(\{\phi\}) = \mathcal{L}$
- Coherency's postulate holds too i.e. $C(\emptyset) \neq \mathcal{L}$ since $C(\emptyset)$ corresponds to all tautologies and, since $\phi = \bot$ is not a tautology then there exists a formula $\phi \in \mathcal{L}$ s.t. $\phi \notin C(\emptyset)$.
- Finally we have also the last postulate $C(\Gamma) = \bigcup_{\Sigma \subseteq \Gamma} C(\Sigma)$.

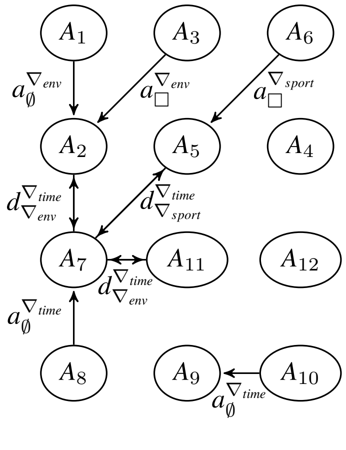

Fig. 3. Argumentation graph

6.4. VAF based on modal logics

We can now define a VAF based on a modal logic, named τ-VAF. In this framework, each conclusion of an argument that is in the form $\star(\phi_1,\dots,\phi_{\rho(\star)})$ is associated with values. For instance, those values may be bound to mental states (truth-seeking for knowledge, egoism for desires, etc.), to idealities (what should be done in the name of a given deontic theory) or moral values (e.g. nullary operators representing the fact that the current world is good, bad or promotes courage, justice, etc.) Let us notice that all arguments that *do not* conclude a formula in the form $\star(\phi_1,\dots,\phi_{\rho(\star)})$ with $\star$ a *non nullary* modality, are associated with the $\emptyset$ value which denotes factual verity e.g. arguments that would conclude p or $\triangle_{p.t.}$ (i.e. "going by public transport").

Definition 18 (AF based on modal logics). *Let $\tau = (\mathcal{O},\rho)$ be a similarity type. We call* τ-AF *a tuple $\langle \mathcal{C}, \langle d,a\rangle, \mathcal{A}, \mathcal{R}\rangle$ where $\mathcal{C}$ is a τ-frame, $\langle d,a\rangle$ is a consistent modal mapping, and $\langle \mathcal{A},\mathcal{R}\rangle$ is an abstract argumentation framework s.t.:*

- *$\mathcal{A}$ contains only valid arguments on $\mathcal{C}$ i.e. valid w.r.t. $\models_{\mathcal{C}}$,*
- *$\mathcal{R} = Reb \cup Und \cup Def \cup \bigcup_{(\star,\star_2)\in(\mathcal{O}^\Omega\cup\{\emptyset\})\times\mathcal{O}^\Omega} Att\langle d,a\rangle_\star^{\star_2}$*

We call a τ-VAF a tuple $\langle \mathcal{C}, \langle d,a\rangle, \mathcal{A}, \mathcal{R}, \mathcal{V}al, \mu, \succ\rangle$ which is a τ-AF where $\langle \mathcal{A}, \mathcal{R}, \mathcal{V}al, \mu, \succ\rangle$ is a VAF.

Example 13. *In our examples, we deduce the following τ-VAF as depicted in Figure 3. Modal attacks are labelled by the type of modal mapping.*

We assume that μ is s.t. for all $\star \in \{\nabla, \triangle\}$:

(1) $\forall\phi,\psi \in \mathcal{L}, \mu(\_,\star_{sport}(\phi,\psi)) = \nu_{health}$
(2) $\forall\phi,\psi \in \mathcal{L}, \mu(\_,\star_{env}(\phi,\psi) = \nu_{env}$
(3) $\forall\phi,\psi \in \mathcal{L}, \mu(\_,\star_{time}(\phi,\psi)) = \nu_{eco}$
(4) $\forall\phi,\psi \in \mathcal{L}, \mu(\_,\star_{gov}(\phi,\psi)) = \nu_{aut}$
(5) $\forall\phi \in \mathcal{L}, \mu(\_,\Box\phi) = \emptyset$
(6) $\forall A \in \mathcal{A}$ s.t. A is not in the form $(\_,\star(\dots))$, $\mu(A) = \emptyset$

Thus, $\mu = \{(A_1,\emptyset), (A_2,\nu_{env}), (A_3,\emptyset), (A_4,\emptyset), (A_5,\nu_{health}), (A_6,\emptyset), (A_7,\nu_{eco}), (A_8,\emptyset), (A_9,\nu_{env}), (A_{10},\emptyset), (A_{11},\nu_{env}), (A_{12},\nu_{aut})\}$. Then, let us assume that the reasoning agent in this example has preferences on values, given by the following transitive relation $\succ$:

$$\emptyset \succ \nu_{health} \succ \nu_{aut} \succ \nu_{env} \succ \nu_{eco}$$

Thus, some arguments such as A_7 are defeated by A_2, A_5, A_{11} which are associated to a preferred value. The VAF-attack is $\{(A_1, A_2)$, (A_3, A_2), (A_6, A_5), (A_5, A_7), (A_2, A_7), (A_{11}, A_7), (A_8, A_7), $(A_{10}, A_9)\}$. *Finally, let us consider the stable extension $\mathcal{E}$:*

$$\mathcal{E} = \{\{A_1, A_3, A_4, A_6, A_8, A_{10}, A_{11}, A_{12}\}\}$$

This result concludes that the user cannot ride a bike since she has no bike. Since there is no congestion and no emergency, the user does not need to take her car to reduce the time of the travel. Furthermore, for the environment it is better to take public transport or walking. However according to A_{12}, since there is a health crisis, it is recommended to not take public transport. Consequently, the ethical decision given by such modelling is to walk.

Even if we do not explicitly model decisions in our VAF, we can formalize them like in [32] where a function $\mathcal{F}_f$ (resp. $\mathcal{F}_c$) assigns to a decision the set of arguments that support it (resp. does not support it). For instance in Example 13, a decision could be acceptable if supported by credulously acceptable arguments i.e. at least one argument supports the decision and belongs to at least one extension.

7. Conclusion

We proposed in this article to model ethical reasoning with a value-based argumentation framework, named τ-VAF, based on a generalized normal modal logic. Such a logic can deal with a set of n-ary modal operators of interest to express moral theories, dyadic deontic operators, mental states or agency operators. We have shown that this logic is strongly sound and strongly complete. Finally, grounding VAF on such a modal logic have led us to propose new kinds of attacks, named *modal attacks*, which generalize standard logical attacks. Indeed, the standard attacks based on logical contradictions are no longer sufficient to catch an intuitive meaning for attacks. To solve this problem, we explicitly introduce *modal mappings* which characterize contradiction between those different kind of operators. Finally, we shown that the standard logic-based attacks have a quasi-symmetry property, i.e. when an argument attacks another, this argument is necessarily attacked by another one. Interestingly, our modal attacks do not have this property, which is relevant to decide a dilemma.

Future works are threefold: modelization perspectives, computational perspectives, and implementation perspectives. Firstly, although we have shown that a basic n-ary modal is sound and complete, we leave abstract the underlying logical system. Hence, a first perspective is to instantiate τ-VAF with a modal logic dedicated to moral agency – such as DL-MA [8] or with a deontic description logic such as proposed in [44] – in order to capture classical ethical problems, e.g. the trolley and footbridge problems. Secondly, we have shown that verifying if a given modal mapping is consistent is NP-complete. However, we should investigate more deeply the complexity of deciding or computing the different parts of τ-VAF. Intuitively, the complexity classes of the argumentation are related to the complexity classes of the underlying logic. Exhibiting fragments, that can provide a trade-off between complexity and expressivity in terms of ethical reasoning, can be of interest. Thirdly, based on interesting logical fragment, it would be of interest to mechanize the whole process, namely implementing a tableau method for the chosen logic to compute the arguments, implementing a method to compute the attacks, and translating the results to use for instance ASPARTIX [45] to find the extensions.

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Appendix A. Soundness, completeness and theorems of GK

In this appendix, we firstly give the proofs of the compactness and soundness of GK. Then we demonstrate that the axiomatic system GK is complete. We show that it verifies the deduction theorems and that it is strongly sound and strongly complete. Finally, we give the proofs of the theorems obtained with the Hilbert system.

A.1. Compactness

Compactness theorems for modal logics allow to write semantic entailment on infinite sets of formulas but also for syntactical GK-deductibility. Theorems 1 and 4 describes the equivalence between an infinite set of modal formulas which is satisfiable, and the fact that there exists a finite subset of these formulas that is also satisfiable. The theorems work for the GK-deductibility and we have the equivalence between an infinite set of formulas that implies a formula and the fact there exists a finite subset that implies a formula.

Theorem 1. *Let Γ be a set of formulas and ϕ be a formula.*
Γ is satisfiable if, and only if, every finite subsets $\Gamma_0 \subseteq_f \Gamma$ are satisfiable.

Proof.
($\Rightarrow$) Let Γ be a set of formulas s.t. Γ is satisfiable and $\Gamma_0 \subseteq \gamma$ be a finite subset of Γ. Thus, there exists a model $\mathcal{M}$ and a world w s.t. $\forall \phi \in \Gamma, \mathcal{M}, w \models \phi$. And obviously, we have for all $\phi \in \Gamma_0$, since $\Gamma_0 \subseteq \Gamma$, $\phi \in \Gamma$. Thus, there exists a model $\mathcal{M}$ and a world w s.t. $\mathcal{M}, w \models \phi$. We just proved that Γ_0 is satisfiable.

($\Leftarrow$) Let us assume a set of formulas Γ s.t. for all finite subsets $\Gamma_0 \subseteq \Gamma$, Γ_0 is satisfiable. Thus, there exists a model $\mathcal{M}_{\Gamma_0}$ and a world w s.t. $\mathcal{M}_{\Gamma_0}, w \models \Gamma_0$. Let define $\mathcal{S} \subseteq 2^\Gamma$ s.t. $\mathcal{S} := \{\Gamma_f : \Gamma_f \subset \Gamma, \Gamma_f \text{ is finite}\}$, and for each $S \in \mathcal{S}$, $F_S := \{S' \in \mathcal{S} : S \subseteq S'\}$. Let define the family $\mathcal{F}$ of all sets F_S with $S \in \mathcal{S}$ i.e. $\mathcal{F} := \{F_S : S \in \mathcal{S}\}$. Obviously, $\mathcal{F}$ is a filter on $\mathcal{S}$. By Tarsky's lemma [46], since every filter can be extended to an ultrafilter, there exists an ultrafilter $\mathcal{U}$ s.t. $\mathcal{F} \subseteq \mathcal{U}$. Let $\phi \in \Gamma$, we have:

- $F_{\{\phi\}} \in \mathcal{U}$
- If $S' \in F_{\{\phi\}}$ then $\phi \in S'$
- If $F_{\{\phi\}} \subseteq \mathcal{S}' = \{S' \in \mathcal{S} | \exists \mathcal{M}_{S'}, \exists w : \mathcal{M}_{S'}, w \models \phi\}$ then $S' \in \mathcal{U}$

By applying the Łoś's Theorem [47][3], we now have that ϕ holds in the ultraproduct $\prod_{S \subseteq \Gamma} \mathcal{M}_S / \mathcal{U}$.

Consequently, $\prod_{S \subseteq \Gamma} \mathcal{M}_S / \mathcal{U}$ satisfies all formulas in Γ.

□

Theorem 4. *Let Γ be a set of formulas and ϕ be a formula.*
$\Gamma \vdash \phi$ if, and only if, there exists a finite subsets $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \phi$.

[3] Despite the proof of the Łoś's Theorem [47] must be adapted for GK, it intuitively follows from the fact that the theorem is proven for first-order logic and that it exists a standard translation for n-ary modal logic to first-order logic [23].

Proof. ($\Rightarrow$) Let us assume $\Gamma \vdash \phi$. Since the GK-deductibility implies the existence of a finite sequence of formulas, we have $\phi_1, \dots, \phi_N \in \Gamma$ s.t. for all $k \in [|1,N|]$, ϕ_k is either an axiom or is deduced from previous formulas by modus ponens. Thus, $\Gamma_0 = \{\phi_1, \dots, \phi_N\}$ is finite and we have $\Gamma_0 \vdash \phi$.

($\Leftarrow$) It directly comes from the augmentation hypothesis i.e. the axiom of PC $\vdash \phi \Rightarrow (\psi \Rightarrow \phi)$. If there exists a finite set of formulas $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \vdash \phi$, then $\vdash \Gamma_0 \Rightarrow \phi$ (by finite deduction theorem 17). Thus, by modus ponens and syllogism, for all $\psi \in \Gamma$, $\Gamma_0 \cup \{\psi\} \vdash \phi$. So, $\Gamma_0 \cup \bigcup_{\psi \in \Gamma} \{\psi\} \vdash \phi$. $\square$

A.2. Soundness

This section aims to prove the soundness of GK. In the sequel we consider any τ-frame $\mathcal{C}_\tau = (\mathcal{W}, \{\mathcal{R}_\triangle\}_{\triangle \in \mathcal{O}})$.

Theorem 5. *Modus ponens, all tautologies and uniform substitution preserves the validity.*

Proof. (Trivial) $\square$

Theorem 6. *The rule of necessitation is valid, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\text{From } \models \psi_k : \models \nabla(\phi_1, \dots, \phi_{k-1}, \psi_k, \phi_{k+1}, \dots, \phi_{\rho(\triangle)}) \qquad (\mathrm{NEC}^k_\nabla)$$

Proof. Let us assume by contraposition that $\not\models \nabla(\phi_1, \dots, \phi_n)$. Thus, there exists $\mathcal{M}$ and $w \in \mathcal{W}$ such that $\mathcal{M}, w \models \neg\nabla(\phi_1, \dots, \phi_k, \dots, \phi_{\rho(\triangle)})$. So, $\mathcal{M}, w \models \triangle(\neg\phi_1, \dots, \phi_k, \neg\phi_{\rho(\triangle)})$. So, there exists $(v_1, \dots, v_n) \in \mathcal{R}_\triangle(w)$, $\forall i \in [|1, \rho(\triangle)|], v_i \models \neg\phi_i$. Thus, $\mathcal{M}, v_k \models \neg\phi_k$. Consequently, $\not\models \phi_k$. $\square$

Theorem 7. *The axioms (K^k_∇) is valid, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\models \nabla(\phi_1, \dots, \phi_k \Rightarrow \psi_k, \dots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \dots, \phi_k, \dots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \dots, \psi_k, \dots, \phi_{\rho(\triangle)}) \qquad (\mathrm{K}^k_\nabla)$$

Proof. Let us assume, by absurd, (K^k_∇) is not valid i.e. $\not\models \nabla(\phi_1, \dots, \phi_k \Rightarrow \psi_k, \dots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \dots, \phi_k, \dots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \dots, \psi_k, \dots, \phi_{\rho(\triangle)})$. Thus, there exists $\mathcal{M}, w$ such that:

$$\mathcal{M}, w \models \nabla(\phi_1, \dots, \phi_k \Rightarrow \psi_k, \dots, \phi_{\rho(\triangle)}) \qquad (1)$$
$$\wedge \nabla(\phi_1, \dots, \phi_k, \dots, \phi_{\rho(\triangle)}) \qquad (2)$$
$$\wedge \neg\nabla(\phi_1, \dots, \psi_k, \dots, \phi_{\rho(\triangle)}) \qquad (3)$$

Thus, by (3), there exists $(v_1, \dots, v_{\rho(\triangle)}) \in \mathcal{R}_\triangle(w), \mathcal{M}, v_1 \models \neg\phi_1, \dots v_k \models \neg\psi_k, \dots v_{\rho(\triangle)} \models \neg\phi_{\rho(\triangle)}$. However, by (1), for all $(x_1, \dots, x_n) \in \mathcal{R}_\triangle(w)$, $\mathcal{M}, x_1 \models \phi_1$, or $\dots$, or $\mathcal{M}, x_k \models \phi_k \Rightarrow \psi_k$, $\dots$, or $\mathcal{M}, x_{\rho(\triangle)} \models \phi_{\rho(\triangle)}$ and since $\forall i \in [|1, \rho(\triangle)|] \setminus \{k\}, \mathcal{M}, v_i \models \neg\phi_i$, we deduce that $\mathcal{M}, v_k \models \neg\psi_k \wedge (\phi_k \Rightarrow \psi_k)$. So, $\mathcal{M}, v_k \models \neg\phi_k$. However, by (2), $\forall (y_1, \dots, y_{\rho(\triangle)}) \in \mathcal{R}_\triangle(w), \exists i \in [|1, \rho(\triangle)|] \setminus \{k\}, \mathcal{M}, y_i \models \phi_i$ or $\mathcal{M}, y_k \models \phi_k$ and since, $\forall j \in [|1, \rho(\triangle)|] \setminus \{k\} \mathcal{M}, v_i \models \neg\phi_j$, we deduce that $\mathcal{M}, v_k \models \phi_k$. Contradiction. $\square$

Theorem 8. *The axiom of duality is valid i.e.:*

$$\models \nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) \Leftrightarrow \neg\triangle(\neg\phi_1, \dots, \neg\phi_{\rho(\triangle)}) \qquad (\mathrm{Dual}_\nabla)$$

Proof. Let us assume, by absurd, $\not\models \nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) \Leftrightarrow \neg\triangle(\neg\phi_1, \dots, \neg\phi_{\rho(\triangle)})$. Thus, there exists $\mathcal{M}, w$ such that $\mathcal{M}, w \models \nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) \wedge \triangle(\neg\phi_1, \dots, \neg\phi_{\rho(\triangle)}) \vee \neg\nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) \wedge \neg\triangle(\neg\phi_1, \dots, \neg\phi_{\rho(\triangle)})$. We need to prove the contradiction for both side of the equivalence :

- $\nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) \wedge \triangle(\neg\phi_1, \dots, \neg\phi_{\rho(\triangle)})$ (1)
- $\neg\nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) \wedge \neg\triangle(\neg\phi_1, \dots, \neg\phi_{\rho(\triangle)})$ (2)

Let assume (1) there exists $(v_1, \ldots, v_{\rho(\triangle)}) \in \mathcal{R}_\triangle(w)$, $\mathcal{M}, v_1 \models \neg\phi, \ldots,$ and $\mathcal{M}, v_{\rho(\triangle)} \models \neg\phi_{\rho(\triangle)}$ and $\forall(x_1, \ldots, x_{\rho(\triangle)}) \in \mathcal{R}_\triangle(w), \mathcal{M}, x_1 \models \phi_1$, ..., or $\mathcal{M}, x_{\rho(\triangle)} \models \phi_{\rho(\triangle)}$. So $\mathcal{M}, v_1 \models \neg\phi_1 \wedge \phi_1$, ..., or $\mathcal{M}, v_{\rho(\triangle)} \models \neg\phi_{\rho(\triangle)} \wedge \phi_{\rho(\triangle)}$.

Let assume (2) it is not the case that (for all $(v_1, \ldots, v_{\rho(\triangle)}) \in \mathcal{R}_\triangle(w)$, $\mathcal{M}, v_1 \models \phi$, or ..., or $\mathcal{M}, v_{\rho(\triangle)} \models \phi_{\rho(\triangle)}$) i.e. there exists $(v_1, \ldots, v_{\rho(\triangle)}) \in \mathcal{R}_\triangle(w)$, $\mathcal{M}, v_1 \models \neg\phi, \ldots,$ and $\mathcal{M}, v_{\rho(\triangle)} \models \neg\phi_{\rho(\triangle)}$.

Furthermore, it is not the case that $(\exists(x_1, \ldots, x_{\rho(\triangle)}) \in \mathcal{R}_\triangle(w), \mathcal{M}, x_1 \models \neg\phi_1$, ..., and $\mathcal{M}, x_{\rho(\triangle)} \models \neg\phi_{\rho(\triangle)})$ i.e. $\forall(x_1, \ldots, x_{\rho(\triangle)}) \in \mathcal{R}_\triangle(w), \mathcal{M}, x_1 \models \phi_1$, ..., or $\mathcal{M}, x_{\rho(\triangle)} \models \phi_{\rho(\triangle)}$. So $\mathcal{M}, v_1 \models \neg\phi_1 \wedge \phi_1$, ..., or $\mathcal{M}, v_{\rho(\triangle)} \models \neg\phi_{\rho(\triangle)} \wedge \phi_{\rho(\triangle)}$.

Consequently, for (1) and (2) we deduce a contradiction. □

Theorem 9. *The system GK is sound.*

Proof. (Trivial) □

Theorem 10. *The normal properties are sound, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\models \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Leftrightarrow (\nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)})) \qquad (NP_\nabla)$$

$$\models \triangle(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow (\triangle(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \triangle(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)})) \qquad (NP_\triangle)$$

Proof. (NP_∇)

($\Rightarrow$) It is obvious by applying the same reasoning as in the proof of Theorem 6 on $\models \phi_k \wedge \psi_k \Rightarrow \phi_k$ and $\models \phi_k \wedge \psi_k \Rightarrow \psi_k$.

($\Leftarrow$) Let us assume by contraposition, there exists $\mathcal{M}, w$ such that:

$$\mathcal{M}, w \models (\nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)})) \wedge \triangle(\neg\phi_1, \ldots, \neg\phi_k \vee \neg\psi_k, \ldots, \phi_{\rho(\triangle)})$$

Since $\mathcal{M}, w \models (\nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \triangle(\neg\phi_1, \ldots, \neg\phi_k \vee \neg\psi_k, \ldots, \phi_{\rho(\triangle)})$, we have $\mathcal{M}, w \models \triangle(\neg\phi_1, \ldots, \neg\psi_k, \ldots, \neg\phi_{\rho(\triangle)})$. But $\mathcal{M}, w \models \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)})$. Contradiction.

$(NP_\triangle)$ The proof is also obvious by applying the same reasoning as the proof of the necessitation soundness on $\models \phi_k \wedge \psi_k \Rightarrow \phi_k$ and $\models \phi_k \wedge \psi_k \Rightarrow \psi_k$. □

A.3. Completeness

In this section we prove the completeness of the logic GK by using maximal consistent sets. Firstly, we define maximal consistent sets for GK. Secondly, we give the proof of the completeness.

A.3.1. Maximal GK-consistent sets

First of all let us recall some well-known results about maximal consistent sets. A set of formulas Σ is *inconsistent* if, and only if, $\exists\psi_1, \ldots, \psi_n \in \Sigma : \vdash \neg \bigwedge_{i=1}^n \psi_i$. A set of formulas Σ is *consistent* if, and only if, Σ is not inconsistent. A set of formulas Γ is a maximal consistent set if, and only if, $\nexists\Gamma' : \Gamma \subsetneq \Gamma', \Gamma'$ consistent. It leads us to the well-known Lindenbaum's lemma: *for all consistent sets of formulas Γ, there exists a set of formulas Γ' s.t $\Gamma \subseteq \Gamma'$ and Γ' is a maximal consistent set* (MCS). Let us consider a MCS Γ and $\phi, \psi \in \mathcal{L}$ two formulas. We have:

(1) MCS1: if $\Gamma \vdash \phi$ then $\phi \in \Gamma$
(2) MCS2: $(\phi \in \Gamma$ or $\neg\phi \in \Gamma)$ and $\neg(\phi \in \Gamma$ and $\neg\phi \in \Gamma)$
(3) MCS3: $(\phi \vee \psi \in \Gamma)$ if, and only if, $\phi \in \Gamma$ or $\psi \in \Gamma$
(4) MCS3': $(\phi \wedge \psi \in \Gamma)$ if, and only if, $\phi \in \Gamma$ and $\psi \in \Gamma$
(5) MCS4: if $[(\phi \Rightarrow \psi \in \Gamma)$ and $(\phi \in \Gamma)]$ then $\psi \in \Gamma$
(6) MCS5: $\vdash \phi$ if, and only if, $\forall\Gamma$ is a MCS , $\phi \in \Gamma$

A.3.2. Canonical model

We give the definition of the canonical model in GK.

Definition 19. *The* canonical model $\mathcal{M}^c = (\mathcal{W}^c, \{\mathcal{R}^c_\triangle\}_{\triangle\in\mathcal{O}}, V^c)$ *for GK is such that :*

- $\mathcal{W}^c$ *is a non-empty set of worlds where each world is a MCS,*
- *For all* $\triangle \in \mathcal{O}, \varrho(\triangle) = (\triangle, \nabla)$*:*

$$\forall w, v_1, \dots, v_{\rho(\triangle)} \in \mathcal{W} : w\mathcal{R}^c(v_1, \dots, v_{\rho(\triangle)}) \text{ if, and only if, } \nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) \in w \Rightarrow \exists j \in [|1, \rho(\triangle)|], \phi_j \in v_j$$

- $V^c : \mathcal{P} \to 2^{\mathcal{W}}$ *is an interpretation function such that* $\forall p \in \mathcal{P}, w \in V^c(p)$ *if, and only if,* $p \in w$*, i.e.:*

$$\forall p \in \mathcal{P}, V^c(p) = \{w | w \in \mathcal{W}^c \wedge p \in w\}$$

We introduce the following notations in order to simplify the proof of the truth lemma.

$$\forall w \in \mathcal{W}^c, \nabla^*(w) := \{(\phi_1, \dots, \phi_{\rho(\triangle)}) | \nabla(\phi_1, \dots, \phi_{\rho(\triangle)}) \in w\}$$

$$\forall w \in \mathcal{W}^c, \forall i \in [|1, \rho(\triangle)|], \nabla^*_i(w) := \{\phi_i | \nabla(\phi_1, \dots, \phi_i, \dots, \phi_{\rho(\triangle)}) \in w\}$$

These notations bring Theorem 16. This theorem means that for all maximum consistent sets w where a formula $\psi = \triangle(\psi_1, \dots, \psi_{\rho(\triangle)}) \in w$, there exists $\rho(\triangle)$ consistent sets $\{S_i\}_{i\in[|1,\rho(\triangle)|]}$ where for all $i \in [|1, \rho(\triangle)|], \psi_i \in S_i$. Furthermore $\{S_i\}_{i\in[|1,\rho(\triangle)|]}$ is such that for all $\nabla(\phi_1, \dots, \phi_j, \dots, \phi_{\rho(\triangle)}) \in w$ at least one $\phi_j \in S_j$. Then by the Lindenbaum's lemma, it implies the existence of a maximal consistent set v_i s.t. $S_i \subseteq v_i$.

Theorem 16. *For all* $w \in \mathcal{W}^c$ *if* $\triangle(\psi_1, \dots, \psi_{\rho(\triangle)}) \in w$ *then there exists* $S_1 \subseteq \nabla^*_1(w), \dots, S_{\rho(\triangle)} \subseteq \nabla^*_{\rho(\triangle)}(w)$, $\forall(\phi_1, \dots, \phi_{\rho(\triangle)}) \in \nabla^*(w), \exists j \in [|1, \rho(\triangle)|], \phi_j \in S_j$ *and* $\forall i \in [|1, \rho(\triangle)|], S_i \cup \{\psi_i\}$ *is GK-consistent..*

Proof. Let $w \in \mathcal{W}^c$ and $\Gamma = \{\nabla(\phi^{(1)}_1, \dots, \phi^{(1)}_{\rho(\triangle)}), \dots, \nabla(\phi^{(M)}_1, \dots, \phi^{(M)}_{\rho(\triangle)}), \dots\} \subseteq w$ be an infinite set of formulas. We define $\forall m \in \mathbb{N}^*, C^{(m)} = \{\phi^{(m)}_k : k \in [|1, \rho(\triangle)|]\}$ be a set of formulas which represents the set and $\mathcal{C} = \{C^{(m)} : m \in [|1, M|]\}$. Let $T = (V, E)$ be the tree defined as:

- $V := \{\phi^{(m)}_k : \nabla(\phi^{(m)}_1, \dots, \phi^{(m)}_k, \dots, \phi^{(m)}_{\rho(\triangle)}) \in \Gamma\} \cup \{\top\}$
- $E := \{(x, y) \in V^2 : \exists k, \exists m, y = \phi^{(m)}_k, \exists path_m = (x_0, \dots, x_{m-1}) \in V^m, x_0 = \top, x_{(m-1)} = x, \forall m' \in [|1, m-1|], \exists k', k'', (x_{(m'-1)} = \phi^{(m'-1)}_{k''}, x_{m'} = \phi^{(m')}_{k'}) \in E, \forall k''', S^{(m-1)}_{k'''} := \{\phi^{(m')}_{k'''} : \phi^{(m')}_{k'''} \in path_m\}$ is GK-consistent, $S^{(m)}_k := S^{(m-1)}_k \cup \{\phi^m_k\}$ is GK-consistent$\}$

We need to check that for all sets of choices (i.e. $\forall C^{(m)} \in \mathcal{C}$), there is at least one choice (i.e. $\exists \phi^{(m)}_k \in C^{(m)}$) s.t. we maintain consistency (i.e. $\exists x_{m-1} \in V, (x_{m-1}, \phi^{(m)}_k) \in E$). Let us prove by recurrence the following statement :

$$\forall C^{(m)} \in \mathcal{C}, \exists \phi^{(m)}_k \in C^{(m)}, \exists x_{m-1} \in V, (x_{m-1}, \phi^{(m)}_k) \in E$$

For $m = 1$, let us assume by absurd that: for all $k \in [|1, \rho(\triangle)|], \phi^{(1)}_k \in C^{(1)}$, $S^{(1)}_k = \{\top\} \cup \{\phi^{(1)}_k\}$ is GK-inconsistent. So, $\forall k \in [|1, \rho(\triangle)|], \vdash \neg(\top \wedge \phi^{(1)}_k)$. So, $\forall k \in [|1, \rho(\triangle)|], \vdash \neg\phi^{(1)}_k \equiv \top$ i.e. $\vdash \phi^{(1)}_k \equiv \bot$. However, obviously, $\vdash \neg\nabla(\bot, \dots, \bot)$ and with (RE_∇), we have $\vdash \neg\nabla(\phi^{(1)}_1, \dots, \phi^{(1)}_{\rho(\triangle)})$, and by (MCS5), we deduce $\neg\nabla(\phi^{(1)}_1, \dots, \phi^{(1)}_{\rho(\triangle)}) \in w$ and by (MCS2) $\nabla(\phi^{(1)}_1, \dots, \phi^{(1)}_{\rho(\triangle)}) \notin w$. Contradiction.

Now, for $m' = m-1$, we assume the following induction hypothesis: $\exists \phi^{(m')}_k \in C^{(m')}, \exists x_{m'-1} \in V, (x_{m'-1}, \phi^{(m')}_k) \in E$.

Let us assume by absurd that: $\forall \phi_k^{(m)} \in C^{(m)}$, $\forall x_{m-1} \in V, (x_{m-1}, \phi_k^{(m)}) \notin E$. Thus, $\forall x_{m-1} \in V$, s.t. $\exists x_{m-2} \in V, (x_{m-2}, \phi_k^{(m-1)}) \in E$, we have $(x_{m-1}, \phi_k^{(m)}) \notin E$. By induction hypothesis we know that there exists at least one $x_{m-1} \in V$, s.t. $\exists x_{m-2} \in V, (x_{m-2}, \phi_k^{(m-1)}) \in E$. And so, there exists $x_0, \ldots, x_{m-1} \in V, x_0 = \top, \forall m' \in [|1, m-1|], \exists k', x_{m'} = \phi_{k'}^{m'}, \forall k, S_k^{(m-1)}$ is GK-consistent. Now let $x_{m-1} \in V$, and $x_{m-2} \in V$, s.t. $(x_{m-2}, \phi_k^{(m-1)}) \in E$. But since $(x_{m-1}, \phi_k^{(m)}) \notin E$, we have, for all $k \in [|1, \rho(\triangle)|]$, there exists $\psi_1, \ldots, \psi_x \in \bigcup_{j\in[|1,m-1|]} S_k^{(m-1)} \cup \{\phi_k^m\}$ s.t. $\vdash \neg \bigwedge_{j\in[|1,x|]} (\psi_j \wedge \phi_k^{(m)})$. Thus, $\vdash \bigwedge_{j\in[|1,x|]} \psi_j \Rightarrow \neg\phi_k^{(m)}$. Let us notice that all ψ_j belongs to at least one $\nabla(\phi_1, \ldots, \psi_j, \ldots, \phi_{\rho(\triangle)}) \in \Gamma_0$ by definition. Thus, by applying (NEC_∇), (K_∇) and modus ponens, we easily show that:

$$\vdash \bigwedge_j \nabla(\phi, \ldots, \psi_j, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \neg\nabla(\phi_1^{(m)}, \ldots, \phi_{\rho(\triangle)}^{(m)})$$

Thus, by MCS5, $\bigwedge_j \nabla(\phi_1^{(j)}, \ldots, \phi_{\rho(\triangle)}^{(j)}) \Rightarrow \neg\nabla(\phi_1^{(m)}, \ldots, \phi_{\rho(\triangle)}^{(m)}) \in w$.

Furthermore, $\bigwedge_j \nabla(\phi_1, \ldots, \psi_j, \ldots, \phi_{\rho(\triangle)}) \in w$. Thus, by MCS4 and MCS2, we deduce $\nabla(\phi_1^{(m)}, \ldots, \phi_{\rho(\triangle)}^{(m)}) \notin w$. Contradiction.

Let us notice that $S_1 = S_1^{(\infty)}, \ldots, S_{\rho(\triangle)} = S_{\rho(\triangle)}^{(\infty)}$ where $\forall k \in [|1, \rho(\triangle)|], S_k^{(\infty)}$ corresponds to the infinite branch of the infinite tree T. This set exists because of the König's lemma (i.e. for all infinite trees with finite branches, there exists an infinite branch) and by definition of the building process we have $\forall(\phi_1, \ldots, \phi_{\rho(\triangle)}) \in \nabla^*(w), \exists j \in [|1, \rho(\triangle)|], \phi_j \in S_j$.

Let us assume by absurd that the sets $S_1 \subseteq \nabla_1^*(w), \ldots, S_{\rho(\triangle)} \subseteq \nabla_{\rho(\triangle)}^*(w)$, $\triangle(\psi_1, \ldots, \psi_{\rho(\triangle)}) \in w$, there exists $i \in [|1, \rho(\triangle)|], S_i \cup \{\psi_i\}$ is GK-inconsistent. Since $S_i \cup \{\psi_i\}$ is GK-inconsistent, there exists $\phi_1, \ldots, \phi_m \in S_i$ such that:

$$\vdash \neg((\bigwedge_{j\in[|1,m|]} \phi_j) \wedge \psi_i)$$

By applying the Hilbert proof system, we have the following deductions:

$$\vdash \bigwedge_{j\in[|1,m|]} \phi_j \Rightarrow \neg\psi_i \tag{Def$_\bot$}$$

$$\vdash \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \bigwedge_{j\in[|1,m|]} \phi_j \Rightarrow \neg\psi_i, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \tag{NEC$^k_\nabla$}$$

$$\vdash \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \bigwedge_{j\in[|1,m|]} \phi_j \Rightarrow \neg\psi_i, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \tag{K$^k_\nabla$}$$
$$\Rightarrow (\nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \bigwedge_{j\in[|1,m|]} \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \Rightarrow \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \neg\psi_i, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}))$$

$$\vdash \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \bigwedge_{j\in[|1,m|]} \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \Rightarrow \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \neg\psi_i, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \tag{MP}$$

$$\vdash (\bigwedge_{j\in[|1,m|]} \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)})) \Rightarrow \tag{NP$^k_\nabla$}$$
$$\nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \neg\psi_i, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)})$$

$$\vdash \neg(\bigwedge_{j\in[|1,m|]} \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \tag{Def$_\bot$}$$

$$\wedge \neg\nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \neg\psi_i, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}))$$

$$\vdash \neg(\bigwedge_{j\in[|1,m|]} \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \wedge \triangle(\psi_1, \ldots, \psi_i, \ldots, \psi_{\rho(\triangle)})) \qquad (\mathrm{Dual}^k_\nabla)$$

If we define $\mathcal{S} = \{\nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)})\}_{j\in[|1,m|]} \cup \{\triangle(\psi_1, \ldots, \psi_i, \ldots, \psi_{\rho(\triangle)}))\}$, then based on the previous Hilbert proof, we deduce that $\mathcal{S}$ is GK-inconsistent. However $\forall j \in \{1, \ldots, m\}$, $(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \in \nabla^*(w)$ if, and only if, $\nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \in w$ and w is a maximal GK-consistent set. Thus, $\bigwedge_{j=1}^n \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \in w$ (by MCS3') and so $\{\nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)})\}_{j\in[|1,m|]}$ is GK-consistent. Hence:

$$\{\nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)})\}_{j\in[|1,m|]} \subseteq w$$

But since $\mathcal{S}$ is GK-inconsistent, we have that $\triangle(\psi_1, \ldots, \psi_i, \ldots, \psi_{\rho(\triangle)}))$ cannot belong to w. Indeed by *reductio ad absurdum*, if we have $\triangle(\psi_1, \ldots, \psi_i, \ldots, \psi_{\rho(\triangle)})) \in w$, we would also have that $\bigwedge_{j=1}^n \nabla(\neg\psi_1, \ldots, \neg\psi_{i-1}, \phi_j, \neg\psi_{i+1}, \ldots, \neg\psi_{\rho(\triangle)}) \wedge \triangle(\psi_1, \ldots, \psi_i, \ldots, \psi_{\rho(\triangle)})) \in w$ (by MCS3'), and $\mathcal{S}$ would be GK-consistent, which is a contradiction. Thus, $\triangle(\psi_1, \ldots, \psi_i, \ldots, \psi_{\rho(\triangle)})) \notin w$.

Consequently we proved that for all $w \in \mathcal{W}^c$ if $\triangle(\psi_1, \ldots, \psi_{\rho(\triangle)}) \in w$ then there exists $S_1 \subseteq \nabla_1^*(w), \ldots, S_{\rho(\triangle)} \subseteq \nabla^*_{\rho(\triangle)}(w)$, $\forall(\phi_1, \ldots, \phi_{\rho(\triangle)}) \in \nabla^*(w), \exists j \in [|1, \rho(\triangle)|], \phi_j \in S_j$ and $\forall i \in [|1, \rho(\triangle)|], S_i \cup \{\psi_i\}$ is GK-consistent..

□

A.3.3. Truth lemma

The following proofs are based on the degree of formulas. We recall the basic notion of a degree of a formula.

Definition 20. *We define* $deg : \mathcal{L} \to \mathbb{N}$ *the degree function if, and only if, for all* $\phi, \psi \in \mathcal{L}$:

(1) $\forall p \in \mathcal{P}, deg(p) = 0$
(2) $deg(\neg\phi) = deg(\phi) + 1$
(3) $deg(\nabla(\phi_1, \ldots, \phi_{\rho(\triangle)})) = max\{deg(\phi_i) : i \in [|1, \rho(\triangle)|]\} + 1$
(4) $deg(\phi \wedge \psi) = max\{deg(\phi), deg(\psi)\} + 1$
(5) $deg(\phi \vee \psi) = max\{deg(\phi), deg(\psi)\} + 1$
(6) $deg(\phi \Rightarrow \psi) = max\{deg(\neg\phi), deg(\psi)\} + 1$

We say that the degree of a formula ϕ *is* $n \in \mathbb{N}^*$ *if, and only if,* $deg(\phi) = n$.

Lemma 2. *Let* $\phi \in \mathcal{L}$ *be a formula, for all* $w \in \mathcal{W}^c$:

$$\mathcal{M}^c, w \models \phi \text{ if, and only if, } \phi \in w$$

Proof. Let us reason by recurrence on the degree of a formula.

(Initialization) If $\phi \in \mathcal{L}$ is a formula such that $deg(\phi) = 0$, i.e. there exists $p \in \mathcal{P}, \phi = p$. By definition of a canonical model we have $\forall w \in \mathcal{W}^c, w \in V(p)$ if, and only if, $p \in w$.

(Heredity) Let us assume that for all formulas $\phi \in \mathcal{L}$ and $n \in \mathbb{N}^*$ such that $deg(\phi) < n$, we have for all $w \in \mathcal{W}^c, \mathcal{M}^c, w \models \phi$ if, and only if, $\phi \in w$.

Let $\psi, \theta \in \mathcal{L}$ such that $max(deg(\psi), deg(\theta)) = n - 1$. So we have for all worlds $w \in \mathcal{W}^c$, $\mathcal{M}^c, w \models \psi$ iff $\psi \in w$ and $\mathcal{M}^c, w \models \theta$ iff $\theta \in w$. Furthermore we have $\mathcal{M}^c, w \models \neg\psi$ iff $\mathcal{M}^c, w \not\models \psi$ iff $\psi \notin w$. Then $\mathcal{M}^c, w \models \psi \wedge \theta$ iff $\mathcal{M}^c, w \models \psi$ and $\mathcal{M}^c, w \models \theta$ iff $\psi \in w$ and $\theta \in w$ iff $\psi \wedge \theta \in w$ (MCS3'). Then $\mathcal{M}^c, w \models \psi \vee \theta$ iff $\mathcal{M}^c, w \models \psi$ or

$\mathcal{M}^c, w \models \theta$ iff $\psi \in w$ or $\theta \in w$ iff $\psi \vee \theta \in w$ (MCS3). Finally $\mathcal{M}^c, w \models \psi \Rightarrow \theta$ iff $\mathcal{M}^c, w \models \neg\psi$ or $\mathcal{M}^c, w \models \theta$ iff $\psi \notin w$ or $\theta \in w$ iff $\psi \Rightarrow \theta \in w$.

Let $\phi_1, \ldots, \phi_{\rho(\triangle)} \in \mathcal{L}$ such that $max(deg(\phi_1), \ldots, deg(\phi_{\rho(\triangle)})) = n - 1$ and $w \in \mathcal{W}^c$.

Let us show that the heredity holds for ∇ modalities by double implication.

($\Rightarrow$) Let us assume by contraposition that $\nabla(\phi_1, \ldots, \phi_{\rho(\triangle)}) \notin w$ and since w is a maximal GK-consistent set, we have $\neg\nabla(\phi_1, \ldots, \phi_{\rho(\triangle)}) \in w$ i.e. $\triangle(\neg\phi_1, \ldots, \neg\phi_{\rho(\triangle)}) \in w$ by applying MCS4 on the duality axiom. By Theorem 16, we have that there exists $S_1 \subseteq \nabla_1^*(w), \ldots, S_{\rho(\triangle)} \subseteq \nabla_{\rho(\triangle)}^*(w)$, $\forall(\phi_1, \ldots, \phi_{\rho(\triangle)}) \in \nabla^*(w), \exists j \in [|1, \rho(\triangle)|], \phi_j \in S_j$ and $\forall i \in [|1, \rho(\triangle)|], S_i \cup \{\psi_i\}$ is GK-consistent.

Thus, for all $i \in [|1, \rho(\triangle)|]$, $S_i \cup \{\neg\phi_i\}$ is GK-consistent and so, by the Lindenbaum's lemma, there exists $v_i \in \mathcal{W}^c : S_i \cup \{\neg\phi_i\} \subseteq v_i$ and v_i is maximal GK-consistent set. So we have $\neg\phi_i \in v_i$ and, by definition of $\mathcal{R}^c$, we have $w\mathcal{R}^c(v_1, \ldots, v_i, \ldots, v_{\rho(\triangle)})$. Furthermore, we have $\phi_i \notin v_i$ and, by induction hypothesis $\mathcal{W}^c, v_i \not\models \phi_i$. So since there exists $v_1, \ldots, v_{\rho(\triangle)} \in \mathcal{W}^c : w\mathcal{R}^c(v_1, \ldots, v_{\rho(\triangle)}) : \forall i \in [|1, \rho(\triangle)|], v_i \models \neg\phi_i$, we have $\mathcal{M}^c, w \models \neg\nabla(\phi_1, \ldots, \phi_{\rho(\triangle)})$, i.e. $\mathcal{M}^c, w \not\models \nabla(\phi_1, \ldots, \phi_{\rho(\triangle)})$.

($\Leftarrow$) By contraposition, let us assume that $\mathcal{M}^c, w \not\models \nabla(\phi_1, \ldots, \phi_{\rho(\triangle)})$, i.e. $\mathcal{M}^c, w \models \neg\nabla(\phi_1, \ldots, \phi_{\rho(\triangle)})$. So there exists $v_1, \ldots, v_{\rho(\triangle)} \in \mathcal{W}^c : w\mathcal{R}^c(v_1, \ldots, v_{\rho(\triangle)}), \forall i \in [|1, \rho(\triangle)|], \mathcal{M}^c, v_i \models \neg\phi_i$. So $\mathcal{M}^c, v_i \not\models \phi_i$ and by induction hypothesis, we have $\phi_i \notin v_i$. However, since $\phi_i \notin v_i$, by definition of $\mathcal{R}^c$, we have $\nabla(\phi_1, \ldots, \phi_{\rho(\triangle)}) \notin w$.

(Conclusion) So we have shown by recurrence that:

$$\forall\phi \in \mathcal{L}, \forall w \in \mathcal{W}^c : \mathcal{M}^c, w \models \phi \text{ if, and only if, } \phi \in w$$

□

Now that the link between validity and GK-consistent maximum sets has been demonstrated, we can prove the link between our canonical model and the proven formulas in our axiomatic system.

Lemma 3 (Truth lemma)**.** *Let $\phi \in \mathcal{L}$ be a formula and $w \in \mathcal{W}^c$,*

$$\mathcal{M}^c, w \models \phi \textit{ if, and only if, } \vdash \phi$$

Proof. Let $\phi \in \mathcal{L}$ be a formula and $w \in \mathcal{W}^c$. We have $\mathcal{M}^c, w \models \phi$ iff (by Lemma 2) $\phi \in w$ iff (by MCS5) $\vdash \phi$. □

We need also an obvious lemma which describes the fact that the canonical model of GK is a GK model.

Lemma 4. *The canonical model $\mathcal{M}^c$ is a GK-model.*

Proof. The proof is obvious by definition of the canonical model and since we do not consider any constraint on the τ-frames. □

Theorem 11. *The GK system is complete.*

Proof. (Completeness) By definition, we have that for all valid formulas ϕ in the τ-frame $\mathcal{C}$, ϕ is valid in all models $\mathcal{M}$ on $\mathcal{C}$. Thus, since the canonical model is a GK model from Lemma 4, ϕ is also valid in the canonical model $\mathcal{M}^c$. So, from the Lemma 3, we have $\vdash \phi$. Consequently, we have proved that the GK system is complete. □

A.4. Deduction theorems

The GK-deductibility has various fundamental properties such as reflexivity, transitivity, and left weakening. We first prove the deduction theorems for the case of finite set of formulas and thanks to the compactness theorems, we extend these results to infinite set of formulas.

Proposition 2. *Let Σ and Γ be two finite sets of formulas in GK. It holds:*

(1) Reflexivity *holds i.e. if $\phi \in \Sigma$, then $\Sigma \vdash \phi$*
(2) Transitivity *holds i.e. if $\Sigma \vdash \phi$ and $\{\phi\} \vdash \psi$, then $\Sigma \vdash \psi$*
(3) Left weakening *holds i.e. if $\Sigma \vdash \phi$ and $\Sigma \subseteq \Gamma$, then $\Gamma \vdash \phi$*

Proof. Let Σ and Γ be two finite non-empty sets of formulas in GK.

(1) If $\phi \in \Sigma$, then since $\vdash \phi \Rightarrow \phi$, by definition of GK-deductibility, we have $\Sigma \vdash \phi$.
(2) If $\Sigma \vdash \phi$ and $\{\phi\} \vdash \psi$, then there exists $\psi_1, \ldots, \psi_n \in \Sigma$ with $n \in \mathbb{N}^*$ such that:

$$\vdash (\bigwedge_{k \in \{1,\ldots,n\}} \psi_k) \Rightarrow \phi$$

Furthermore since $\{\phi\} \vdash \psi$, by definition we have $\vdash \phi \Rightarrow \psi$. Thus, we deduce:

$$\vdash (\bigwedge_{k \in \{1,\ldots,n\}} \psi_k) \Rightarrow \psi$$

Consequently, we prove that $\Sigma \vdash \psi$.
(3) If $\Sigma \vdash \phi$ and $\Sigma \subseteq \Gamma$, then there exists $\psi_1, \ldots, \psi_n \in \Sigma$ with $n \in \mathbb{N}^*$ such that:

$$\vdash (\bigwedge_{k \in \{1,\ldots,n\}} \psi_k) \Rightarrow \phi$$

Since $\psi_1, \ldots, \psi_n \in \Sigma$ and $\Sigma \subseteq \Gamma$, we have $\psi_1, \ldots, \psi_n \in \Gamma$. Thus, we prove that $\Gamma \vdash \phi$.

□

We prove now the deduction theorems of GK. We first recall the definition of a model of a set of formula.

Definition 21. *Let Σ be a set of formulas and ϕ be a formula. Σ* semantically entails *ϕ, written $\Sigma \models \phi$ if, and only if, for all models $\mathcal{M}$ of Σ (i.e. $\forall \psi \in \Sigma, \mathcal{M} \models \psi$), $\mathcal{M} \models \phi$.*

Theorem 17. *Let Γ be a finite set of formulas and ϕ be a formula. GK has deduction theorems (1) and (2):*

(1) $\Gamma \vdash \phi$ iff $\vdash \Gamma \Rightarrow \phi$ (Syntactical form)
(2) $\Gamma \models \phi$ iff $\models \Gamma \Rightarrow \phi$ (Semantics form)

Proof. Let $\Gamma \subseteq \mathcal{L}$ be a finite set of formulas, ϕ and ψ be two formulas of $\mathcal{L}$. Let us show the syntactical form of the deduction theorem (1).

($\Rightarrow$) Let us assume that $\Gamma \cup \{\psi\} \vdash \phi$. By definition of the GK-deductibility, we have that there exists $\Sigma = \{\psi_1, \ldots, \psi_n\}, \Sigma \subseteq \Gamma \cup \{\psi\}$ such that:

$$\vdash \bigwedge_{i \in \{1,\ldots,n\}} \psi_i \Rightarrow \phi$$

We have two cases to consider $\psi \in \Sigma$ and when $\psi \notin \Sigma$.

a) If $\psi \in \Sigma$, then there exists $i \in \{1,\ldots,n\}$ such that $\psi = \psi_i$. So $\vdash (\bigwedge_{k\in\{1,\ldots,n\}\setminus\{i\}} \psi_k) \wedge \psi \Rightarrow \phi$ by commutativity and by associativity of $\wedge$. Then, $\vdash \bigwedge_{k\in\{1,\ldots,n\}\setminus\{i\}} \psi_k \Rightarrow (\psi \Rightarrow \phi)$. However for all $k \in \{1,\ldots,n\} \setminus \{i\}, \psi_k \in \Sigma$ and so $\psi_k \in \Gamma$ by inclusion. Consequently, thus $\Gamma \vdash \psi \Rightarrow \phi$.

b) If $\psi \notin \Sigma$, then for all $i \in \{1,\ldots,n\}, \psi_i \neq \psi$. We have

$$\vdash \bigwedge_{k\in\{1,\ldots,n\}} \psi_k \Rightarrow \phi$$

Or since $\vdash \phi \Rightarrow (\psi \Rightarrow \phi)$ is a PC axiom, we immediately deduce:

$$\vdash \bigwedge_{k\in\{1,\ldots,n\}} \psi_k \Rightarrow (\psi \Rightarrow \phi)$$

Consequently, since for all $k \in \{1,\ldots,n\}, \psi_k \in \Sigma$ and so $\psi_k \in \Gamma$ by inclusion. Thus, $\Gamma \vdash \psi \Rightarrow \phi$.

So, we prove that if $\Gamma \cup \{\psi\} \vdash \phi$ then $\Gamma \vdash \psi \Rightarrow \phi$.

($\Leftarrow$) Let us assume that $\Gamma \vdash \psi \Rightarrow \phi$. So there exists $\Sigma = \{\psi_1,\ldots,\psi_n\}$, $\Sigma \subseteq \Gamma$ such that:

$$\vdash \bigwedge_{i\in\{1,\ldots,n\}} \psi_i \Rightarrow (\psi \Rightarrow \phi)$$

Thus,

$$\vdash \bigwedge_{i\in\{1,\ldots,n\}} \psi_i \wedge \psi \Rightarrow \phi$$

is a theorem. So we deduce that $\Sigma \cup \{\psi\} \vdash \phi$. But since $\Sigma \subseteq \Gamma$, by left weakening, we immediately deduce that $\Gamma \cup \{\psi\} \vdash \phi$. Thus, if $\Gamma \vdash \psi \Rightarrow \phi$ then $\Gamma \cup \{\psi\} \vdash \phi$.

Conclusion, we prove that $\Gamma \vdash \phi$ iff $\vdash \Gamma \Rightarrow \phi$.

The semantic version of the deduction theorem (2) immediately follows:

$\Gamma \models \psi \Rightarrow \phi$ iff ($\forall \mathcal{M}$: if $\mathcal{M} \models \Gamma$, then $\mathcal{M} \models \psi \Rightarrow \phi$) iff $\forall \mathcal{M} : \mathcal{M} \models \Gamma \Rightarrow (\psi \Rightarrow \phi)$ iff $\forall \mathcal{M} : \mathcal{M} \models \neg\Gamma \vee \neg\psi \vee \phi$ iff $\forall \mathcal{M} : \mathcal{M} \models \neg(\Gamma \wedge \psi) \vee \phi$ iff $\forall \mathcal{M} : \mathcal{M} \models (\Gamma \wedge \psi) \Rightarrow \phi$ iff ($\forall \mathcal{M}$: if $\mathcal{M} \models \Gamma \cup \{\psi\}$, then $\mathcal{M} \models \phi$) iff $\Gamma \cup \{\psi\} \models \phi$. $\square$

Thanks to the compactness Theorem 4 on GK-deductibility all previous results hold for infinite sets of formulas.

Theorem 12. *Let Γ be an infinite set of formulas and ϕ be a formula. GK has the infinite deduction theorems (1) and (2):*

(1) $\Gamma \vdash \phi$ iff $\vdash \Gamma \Rightarrow \phi$ (Syntactical form)
(2) $\Gamma \models \phi$ iff $\models \Gamma \Rightarrow \phi$ (Semantics form)

Proof. Let Γ be an infinite set of formulas and ϕ be a formula,

(1) ($\Rightarrow$) If $\Gamma \vdash \phi$, thus there exists $\Gamma_0 \subseteq_f \Gamma$ by the compactness theorem 4 s.t. $\Gamma_0 \vdash \phi$, thus by the finite deduction theorem 17, we have $\vdash \Gamma_0 \Rightarrow \phi$. Then, by notation since Γ_0 is finite and s.t. $\Gamma_0 \subseteq \Gamma$, and because we have the compactness theorem, we allow the notation $\vdash \Gamma \Rightarrow \phi$.

($\Leftarrow$) If $\vdash \Gamma \Rightarrow \phi$, by notation, it means, there exists $\Gamma_0 \subseteq_f \Gamma$ s.t. $\vdash \Gamma_0 \Rightarrow \phi$. By deduction theorem, we have $\Gamma_0 \vdash \phi$ and thus, by compactness theorem 4 we have $\Gamma \vdash \phi$.

(2) We have the following proof :

$$\begin{aligned}
&\Gamma \models \phi && (1)\\
&\text{iff } \forall \mathcal{M},\ \text{if } \mathcal{M} \models \Gamma \text{ then } \mathcal{M} \models \phi && (2)\\
&\text{iff } \forall \mathcal{M},\ \text{if } \forall \Gamma_0 \subseteq_f \Gamma, \mathcal{M} \models \Gamma_0 \text{ then } \mathcal{M} \models \phi && (3)\\
&\text{iff } \bigcup_{\Gamma_0 \subseteq_f \Gamma} \Gamma_0 \models \phi && (4)\\
&\text{iff } \exists \Gamma_0 \subseteq_f \Gamma, \Gamma_0 \models \phi && (5)\\
&\text{iff } \exists \Gamma_0 \subseteq_f \Gamma, \models \Gamma_0 \Rightarrow \phi && (6)\\
&\text{iff } \models \bigvee_{\Gamma_0 \subseteq_f \Gamma} (\Gamma_0 \Rightarrow \phi) && (7)\\
&\text{iff } \models (\bigwedge_{\Gamma_0 \subseteq_f \Gamma} \Gamma_0) \Rightarrow \phi && (8)\\
&\text{iff } \models \Gamma \Rightarrow \phi && (9)
\end{aligned}$$

□

A.5. Strong soundness

We demonstrate now the strong soundness of GK. The strong soundness is an almost immediate consequence of the soundness and the semantic weakening that we demonstrated previously.

Theorem 13 (Strong soundness of GK)**.** *Let Γ be a set of formulas of GK, we have that the system GK is strongly sound, i.e. if $\Gamma \vdash \phi$ then $\Gamma \models \phi$.*

Proof. Let Γ be a set of formulas of GK, ϕ be a formula of GK. Let us assume that $\Gamma \vdash \phi$, so there exists $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash \psi_1 \wedge \ldots \wedge \psi_n \Rightarrow \phi$. Thus, by soundness we have $\models \psi_1 \wedge \ldots \wedge \psi_n \Rightarrow \phi$ i.e. $\models \neg\psi_1 \vee \ldots \vee \neg\psi_n \vee \phi$. So:

$$\models \neg\psi_1 \vee \ldots \vee \neg\psi_n \vee \neg \bigwedge_{\theta \in \Gamma} \theta \vee \phi$$

Thus, by applying the rule of De Morgan, we have:

$$\models \neg \bigwedge_{\theta \in \Gamma \cup \{\psi_1, \ldots, \psi_n\}} \theta \vee \phi$$

However since $\{\psi_1, \ldots, \psi_n\} \subseteq \Gamma$, we have $\Gamma \cup \{\psi_1, \ldots, \psi_n\} = \Gamma$ and so (semantic weakening):

$$\models \neg \bigwedge_{\theta \in \Gamma} \theta \vee \phi$$

Consequently, we have $\forall \mathcal{M},\ \text{if } \mathcal{M} \models \Gamma,\ \text{then } \mathcal{M} \models \phi$. Thus, we have proved that $\Gamma \models \phi$. □

A.6. Strong completeness

The GK system is strongly complete. In order to demonstrate the strong completeness of the system, we need the next lemma, denoted Lemma 5. This lemma means that for all consistent sets Γ, there exists a world w in the canonical model where all $\phi \in \Gamma, \mathcal{W}^c, w \models \phi$.

Lemma 5. *For all GK-consistent sets Γ of formulas, there exists a world $w \in \mathcal{W}^c$ in the canonical model $\mathcal{M}^c$ such that $\mathcal{M}^c, w \models \Gamma$, i.e. $\forall \phi \in \Gamma : \mathcal{M}^c, w \models \phi$.*

Proof. Let $\mathcal{M}^c$ be the canonical model. Let Γ be a GK-consistent set of formulas. By applying the lemma of Lindenbaum, there exists a maximal consistent set of formulas Γ' such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \in \mathcal{W}^c$. Let $w = \Gamma'$ denotes the possible world in $\mathcal{W}^c$. We have $\forall \phi \in \Gamma' : \mathcal{M}^c, \Gamma' \models \phi$, and so $\forall \phi \in \Gamma : \mathcal{M}^c, \Gamma \models \phi$ i.e. $\forall \phi \in \Gamma : \mathcal{M}^c, w \models \phi$. Thus, we have proved that for all GK-consistent sets Γ, there exists a world $w \in \mathcal{W}^c$ satisfying all formulas of Γ in the canonical model $\mathcal{M}^c$. $\square$

Finally we give the proof of the strong completeness of GK.

Theorem 14 (Strong completeness of GK)**.** *Let $\phi \in \mathcal{L}$ be a formula. For all sets $\Gamma \subseteq \mathcal{L}$ of formulas, we have that the system GK is strongly complete i.e. if $\Gamma \models \phi$, then $\Gamma \vdash \phi$*

Proof. By contraposition, let $\Gamma \subseteq \mathcal{L}$ be a set of formulas such that $\Gamma \not\vdash \phi$, we have that $\Gamma \cup \{\neg\phi\}$ is a GK-consistent set. Indeed, by absurdity, if we have $\Gamma \cup \{\neg\phi\}$ is GK-inconsistent, then we would have that there exists $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash \neg(\psi_1 \wedge \ldots \wedge \psi_n \wedge \neg\phi)$, and so we would also have $\vdash (\psi_1 \wedge \ldots \wedge \psi_n) \Rightarrow \phi$. However by deduction theorem, we would deduce that $\Gamma \cup \{\psi_1, \ldots, \psi_n\} \vdash \phi$, i.e. $\Gamma \vdash \phi$, which contradicts the hypothesis $\Gamma \not\vdash \phi$. Thus, by lemma 5, there exists a model $\mathcal{M}$ (the canonical model) and a world w such that $\mathcal{M}, w \models \Gamma \cup \{\neg\phi\}$, i.e. $\mathcal{M}, w \models \Gamma$ and $\mathcal{M}, w \models \neg\phi$. Consequently we have proved that there exists a model $\mathcal{M}$ such that $\mathcal{M}, \Gamma \not\models \phi$. $\square$

A.7. Theorems of GK

In this section we give the proof of the generalized normal properties.

Theorem 2. *The rule of equivalence is verified, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\text{From } \vdash \phi_k \Leftrightarrow \psi_k : \vdash \nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)})) \Leftrightarrow \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)}) \qquad (\mathrm{RE}^k_\nabla)$$

Proof. ($\mathrm{RE}^k_\star$)
For a sake of readability and because the theorem is obvious, we give here the steps of the proof. First, considering that $(\phi_k \Leftrightarrow \psi_k) \equiv (\phi_k \Rightarrow \psi_k \wedge \psi_k \Rightarrow \phi_k)$, then applying ($\mathrm{NEC}^k_\nabla$) on $\phi_k \Leftrightarrow \psi_k$, then applying ($\mathrm{K}^k_\nabla$) on each side of the equivalence, and finally applying modus ponens let possible to deduce the theorem for ∇ modalities. $\square$

Theorem 3. *The normal properties hold, i.e. for all $k \in [|1, \rho(\triangle)|]$:*

$$\vdash \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Leftrightarrow (\nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)})) \qquad (NP^k_\nabla)$$

$$\vdash \triangle(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow (\triangle(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \triangle(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)})) \qquad (NP^k_\triangle)$$

Proof.

$(NP^k_{\bigtriangledown})$
$(\Rightarrow)$

$$\vdash \phi_k \wedge \psi_k \Rightarrow \phi_k \quad (\text{Elim}_\wedge)$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k \Rightarrow \phi_k, \ldots, \phi_{\rho(\triangle)}) \quad (\text{NEC}^k_{\bigtriangledown})$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k \Rightarrow \phi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \quad (\text{K}^k_{\bigtriangledown})$$
$$(\nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}))$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \quad (\text{MP})$$
$$\vdash \phi_k \wedge \psi_k \Rightarrow \psi_k \quad (\text{Elim}_\wedge)$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k \Rightarrow \psi_k, \ldots, \phi_{\rho(\triangle)}) \quad (\text{NEC}^k_{\bigtriangledown})$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k \Rightarrow \psi_k, \ldots, \phi_{\rho(\triangle)}) \quad (\text{K}^k_{\bigtriangledown})$$
$$\Rightarrow (\nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)}))$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)}) \quad (\text{MP})$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow (\nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)})) \quad (\text{Intro}_\wedge)$$

$(\Leftarrow)$

$$\vdash \phi_k \Rightarrow (\psi_k \Rightarrow \phi_k \wedge \psi_k) \quad (\text{Intro}_\wedge)$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k \Rightarrow (\psi_k \Rightarrow \phi_k \wedge \psi_k), \ldots, \phi_{\rho(\triangle)}) \quad (\text{NEC}^k_{\bigtriangledown})$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k \Rightarrow (\psi_k \Rightarrow \phi_k \wedge \psi_k), \ldots, \phi_{\rho(\triangle)}) \Rightarrow \quad (\text{K}^k_{\bigtriangledown})$$
$$(\nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \psi_k \Rightarrow \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}))$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \psi_k \Rightarrow \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \quad (\text{MP})$$
$$\vdash \nabla(\phi_1, \ldots, \psi_k \Rightarrow \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \quad (\text{K}^k_{\bigtriangledown})$$

By applying the syllogism and modus ponens on the previous theorems, we deduce the following theorem:

$$\vdash \nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow (\nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \quad (\text{MP})$$
$$\vdash \nabla(\phi_1, \ldots, \phi_k, \ldots, \phi_{\rho(\triangle)}) \wedge \nabla(\phi_1, \ldots, \psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \phi_k \wedge \psi_k, \ldots, \phi_{\rho(\triangle)}) \quad (\text{MP})$$

$(NP^k_{\triangle})$

$$\vdash \phi_k \wedge \psi_k \Rightarrow \psi_k \quad (\text{Elim}_\wedge)$$
$$\vdash \neg\psi_k \Rightarrow \neg(\phi_k \wedge \psi_k) \quad (\text{Contrap.})$$
$$\vdash \nabla(\phi_1, \ldots, \neg\psi_k \Rightarrow \neg(\phi_k \wedge \psi_k), \ldots, \phi_{\rho(\triangle)}) \quad (\text{NEC}^k_{\bigtriangledown})$$
$$\vdash \nabla(\phi_1, \ldots, \neg\psi_k \Rightarrow \neg(\phi_k \wedge \psi_k), \ldots, \phi_{\rho(\triangle)}) \quad (\text{K}^k_{\bigtriangledown})$$
$$\Rightarrow (\nabla(\phi_1, \ldots, \neg\psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \neg(\phi_k \wedge \psi_k), \ldots, \phi_{\rho(\triangle)}))$$
$$\vdash \nabla(\phi_1, \ldots, \neg\psi_k, \ldots, \phi_{\rho(\triangle)}) \Rightarrow \nabla(\phi_1, \ldots, \neg(\phi_k \wedge \psi_k), \ldots, \phi_{\rho(\triangle)}) \quad (\text{MP})$$

$$\vdash \neg\nabla(\phi_1,\ldots,\neg(\phi_k \wedge \psi_k),\ldots,\phi_{\rho(\triangle)}) \Rightarrow \neg\nabla(\phi_1,\ldots,\neg\psi_k,\ldots,\phi_{\rho(\triangle)}) \qquad \text{(Contrap.)}$$

$$\vdash \triangle(\neg\phi_1,\ldots,\phi_k \wedge \psi_k,\ldots,\neg\phi_{\rho(\triangle)}) \Rightarrow \triangle(\neg\phi_1,\ldots,\psi_k,\ldots,\neg\phi_{\rho(\triangle)}) \qquad (\text{Dual}_\nabla)$$

$$\vdash \triangle(\phi_1,\ldots,\phi_k \wedge \psi_k,\ldots,\phi_{\rho(\triangle)}) \Rightarrow \triangle(\phi_1,\ldots,\psi_k,\ldots,\phi_{\rho(\triangle)}) \qquad (\text{USub: } \forall i \in [|1,\rho(\triangle)|] \setminus \{k\}, \phi_i \mapsto \neg\phi_i)$$

By applying the left conjunction elimination, we deduce the following theorem:

$$\vdash \triangle(\phi_1,\ldots,\phi_k \wedge \psi_k,\ldots,\phi_{\rho(\triangle)}) \Rightarrow \triangle(\phi_1,\ldots,\phi_k,\ldots,\phi_{\rho(\triangle)}) \qquad \text{(MP)}$$

Finally, we obtain the desired result by applying the modus ponens on the conjunction introduction:

$$\begin{aligned}
&\vdash [\triangle(\phi_1,\ldots,\phi_k \wedge \psi_k,\ldots,\phi_{\rho(\triangle)}) \Rightarrow \triangle(\phi_1,\ldots,\psi_k,\ldots,\phi_{\rho(\triangle)})] && \text{(Syll.)}\\
&\quad \Rightarrow [(\triangle(\phi_1,\ldots,\phi_k \wedge \psi_k,\ldots,\phi_{\rho(\triangle)}) \Rightarrow \triangle(\phi_1,\ldots,\phi_k,\ldots,\phi_{\rho(\triangle)}))\\
&\quad \Rightarrow [\triangle(\phi_1,\ldots,\phi_k \wedge \psi_k,\ldots,\phi_{\rho(\triangle)}) \Rightarrow (\triangle(\phi_1,\ldots,\phi_k,\ldots,\phi_{\rho(\triangle)}) \wedge \triangle(\phi_1,\ldots,\psi_k,\ldots,\phi_{\rho(\triangle)}))]]\\
&\vdash \triangle(\phi_1,\ldots,\phi_k \wedge \psi_k,\ldots,\phi_{\rho(\triangle)}) \Rightarrow (\triangle(\phi_1,\ldots,\phi_k,\ldots,\phi_{\rho(\triangle)}) \wedge \triangle(\phi_1,\ldots,\psi_k,\ldots,\phi_{\rho(\triangle)})) && \text{(MP)}
\end{aligned}$$

□